Measurement 125 (2018) 659–666

Contents lists available at ScienceDirect

Measurement

journal homepage: www.elsevier.com/locate/measurement

The biased balance: Observation, formalism and interpretation of a dissymmetric measuring device

Marc Le Menestrel

Universitat Pompeu Fabra, Departament d'Economia i Empresa, Ramon Trias Fargas 25-27, 08005 Barcelona, España, Spain

ARTICLE INFO

Keywords:
Representational theory of measurement
Biased measurement
Bias
Rationality
Irrationality
Observer
Dependence
Indeterminacy
Invariance

ABSTRACT

This paper studies a balance whose unobservable fulcrum is not necessarily located at the middle of its two pans. It presents three different models, showing how this lack of symmetry modifies the observation, the formalism and the interpretation of such a biased measuring device. It argues that the biased balance can be an interesting source of inspiration for our abstract understanding of how a measuring device influences the measurement process.

> *Then, at last, as they were nearing the fountains for the fourth time, the father of all balanced his golden scales and placed a doom in each of them, one for Achilles and the other for Hector. As he held the scales by the middle, the doom of Hector fell down deep into the house of Hades-and then Phoebus Apollo left him.*
> Homer, *Iliad* XXII.
>
> *Give me a place to stand on, and I can move the earth.*
> Archimedes

1. Introduction

What would have happened had Apollo not taken his scales by the middle? Depending on what we assume to observe with such a biased measuring device, how can we formalize empirical observation and how can we interpret the numbers issued from measurement? This paper proposes a rigorous study of these questions in the context of a scale, or balance, that is not necessarily composed of arms of equal lengths.

A main motivation for broadening our understanding of measurement with the study of a biased balance lies in the universality of the unbiased balance for measurement and judgment. Osiris uses a balance to measure the soul of the dead in ancient Egypt. In the Greek epic tradition, deities like Apollo use a balance to decide of the fate of heroes. As a measuring device, it is discussed by Plato, Aristotle, Euclid and Archimedes [11,24]. It appears in the Bible as a symbol for rigor and exactness and in the Koran as a symbol of supreme wisdom. It symbolizes the invariable middle in ancient China, is part of the Sanskrit mythology and of the Indian and Tibetan spiritual traditions [5]. In the middle ages, the balance was essential to evaluate the price of goods and to allow for the development of trade [13]. Nowadays, it is a symbol of justice all over the modern world. It is ubiquitous in the philosophy of science [3,8,4] and is a seminal example for the foundations of measurement (e.g. [9,27,25]). Historically, the equal-arm balance has been a model for the measurement of objects and for the intuition of unbiased judgment. By studying a biased balance, we intend to better understand how measurement is affected by a biased measuring device and how biased judgements may be modelled.

This is especially true for the representational theory of measurement [9]. This abstract approach to the foundations of measurement formulates formal axioms that can describe empirical observation and be necessary and sufficient to prove the existence and uniqueness of a measuring scale. Let us show how this works with an equal-arm balance. Suppose we position an object, denoted x, on one of its pan and an object y on the other pan. Suppose that we observe that *x is lower than y*. This observation is formally described with a binary relation $\succ_0$ as "$x \succ_0 y$". Adding another object z to x,we observe that *x with z are lower than y*. Since this happens for any object z, the empirical regularity of such an observation leads to assume the following property:

$$\text{for all } x,y,z\colon\ x \succ_0 y \Rightarrow (x \circ z) \succ_0 y,$$

where "$\circ$" naturally means the operation of jointly positioning two objects on the same pan of the balance. Further axioms then reflect the laws or regularities that can be observed, including in particular the

E-mail address: marc.lemenestrel@upf.edu.

https://doi.org/10.1016/j.measurement.2018.05.009
Received 15 December 2014; Received in revised form 11 April 2018; Accepted 2 May 2018
Available online 05 May 2018
0263-2241/

following *additive independence* property:

for all x,y,z: $x \succ_0 y \Leftrightarrow (x \circ z) \succ_0 (y \circ z)$.

With sufficient axioms characterizing such an abstract and idealized setting (the measurement is performed in a locally uniform gravitational field, there is no uncertainty nor any other influence on the measuring process, etc.), the task of representational measurement is then to prove the existence of a function, say φ, which assigns a number to each object such that an object is lower than another on the balance if and only if it is assigned a greater number. Formally, we prove that there exists a real-valued function φ such that

$$x \succ_0 y \Leftrightarrow \varphi(x) > \varphi(y),$$

$$\varphi(x \circ y) = \varphi(x) + \varphi(y).$$

Such a representation theorem builds on Hölder's theorem (see [23] for an English translation) and the theory of extensive measurement (see [9, Chapter 3]). In such an abstract and idealized setting and because the balance is assumed to be of equal arms, the number $\varphi(x)$ can be interpreted as the mass of x as in classical mechanics. The function φ is unique up to multiplication by a positive constant and is called a ratio-scale (see [31]). Systematic predictions can be justified by this formalization. For instance, if the sum of the mass of y and the mass of z is greater than the mass of x, we predict with certainty that we will observe that *y with z is lower than x*. With this abstract and idealized model of the equal-arm balance (including the assumptions of a locally uniform gravitational field, etc.), the observed relation between objects does not depend on the measuring device and its formalization does not depend on the observer. Also, the observed empirical relation is formalized with formal (non-numerical) statements which univocally correspond with observation. Finally, a numerical representation is provided which measures objects and the function alone suffices to this measurement. Things are different with the biased balance. Depending on what we observe, the bias may induce less empirical regularities that must be reflected with weaker, and thus more general, axioms. A first question arises as whether we can still measure objects with a ratio-scale. Another question is whether we can measure the bias of the balance and if yes, what does that measure means.

A biased balance is a two-arm balance whose fulcrum is not necessarily located at the middle of the two pans. The principle of the balance with unequal arms as a measurement of torque has long been understood, at least since Archimedes' proof of the principle of the lever (Propositions 6 and 7 of Book I of *On the equilibrium of the planes*, see [11, p. 192]). Also, the so-called Roman or Steelyard balance, where objects positioned on a tray at one end of the beam are balanced by moving a counterweight along the opposite side of the beam, has been employed to weigh large bodies from the earliest time. Not only the principle of the lever had to be invoked, but also the account of the weight of the tray (or hook) used to hold the object to be weighed, which induces some complications (see for instance the *Liber de Canonio* in [24]). As shown in Suppes [32], these earlier mathematical approaches are very close to the contemporaneous theory of conjoint measurement [9, Chapter 6]. What they share in particular is that they start with two quantities (here weights and distances) which can be manipulated independently in order to observe their conjoint effect. In particular, it is assumed possible to select the distances from the fulcrum so that they are of appropriate proportions. Also, it is assumed that distances can be divided into segments of equal length. In this manner, we can use the device to measure torque and from the measurement of distance derive an indirect measurement of weights.

Our study of the biased balance is of interest and novelty because it does not assume that the distance from the fulcrum is an observable primitive. Depending on what we observe as a relation among objects, we characterize the implicit role of the bias. Hence, we do not start from two quantities playing similar roles but with one that is observable (the objects positioned on the balance and acting on it with their mass) and infer the role of a factor that is not directly observable (the bias of the balance). Because of the hidden role of the bias, the relation between objects presents less regularity. Therefore, we need to relax some of the properties of the axioms that are supposed to describe the empirical observation of an equal-arm balance. The biased balance being a form of generalization of the equal-arm balance (that the fulcrum is located in the middle is a special case), it provides a model as to how the representational theory of measurement can be generalized. For instance, the representational theory of measurement treats axioms such as completeness and transitivity as necessary for the existence of a ratio-scale.[1] The theory of biased measurement [15–17,14] shows how we can derive the existence of a ratio-scale while relaxing such axioms.

In order to study different assumptions about what can be observed and different formalizations of empirical observation, this paper presents three models of the biased balance. Each model assumes different ways to observe the behavior of the biased balance. Thus, each model shows distinct empirical regularities which are reflected in different set of axioms. Each set of axioms leads to a representation theorem proving that, even with the irregularities emanating from the bias of the balance, a ratio-scale measure of the mass of objects can be shown to exist. These theorems also reveal a numerical factor which quantifies these irregularities and which intuitively corresponds to the bias of the balance. The interpretation of such number is not necessarily obvious, and we make precise what it means and what it quantifies. The biased balance hence leads to a more detailed analysis of the correspondence between empirical observation and its formalization as a relational structure. This step is usually taken for granted in the theory of representational measurement, due to an implicit assumption of the symmetry of the measuring device. Methodologically, we study the biased balance following three fundamental questions:

1. What do we observe and how can we formally describe it?
2. What numerical representation can be constructed from this formal description?
3. What is the meaning of the numbers that we have constructed?

These questions are essential to a clear and precise understanding of the use of numbers and of mathematical models in sciences. Because the biased balance shows how the theory of representational measurement may be broadened to apply to phenomena which do not present the typical empirical regularities assumed by the symmetry of the measuring device, it contributes to address one of its most interesting critiques (e.g. [28,20,21,2]).

The rest of the paper is structured as follows. In Section 2, we present the basic terms and formal properties that we use to study a biased balance. We also introduce the 3 models. In Section 3, we present the first model which assumes that we can observe on which arm of the balance each object is positioned. This model is the closest to the intuition that the biased balance measures the torque and that a conjoint approach should allow to measure both the mass and the distance that compose it. This is carried out by formally defining "extended objects" that are composed of an object together with the arm on which it lies. In that case, we show how we can conjointly construct a ratio-scale that measures the mass of objects and, for each biased balance, a unique pair of numbers that measures the distances between the fulcrum and each pan. In Section 4, a second model assumes that we can observe whether a given object is positioned on the left or on the right from the observer perspective. We do not however define "extended objects" and let the formalism implicitly reflect the left and right distinction that depends on the observer. We show that this corresponds to the most general mathematical properties but still allows for a ratio-scale measuring objects to be constructed. Further, we show that we can

[1] Mathematically, any representation of the form $x \succsim y \Leftrightarrow \varphi(x) \geqslant \varphi(y)$ must assume that the relation $\succsim$ is complete and transitive.

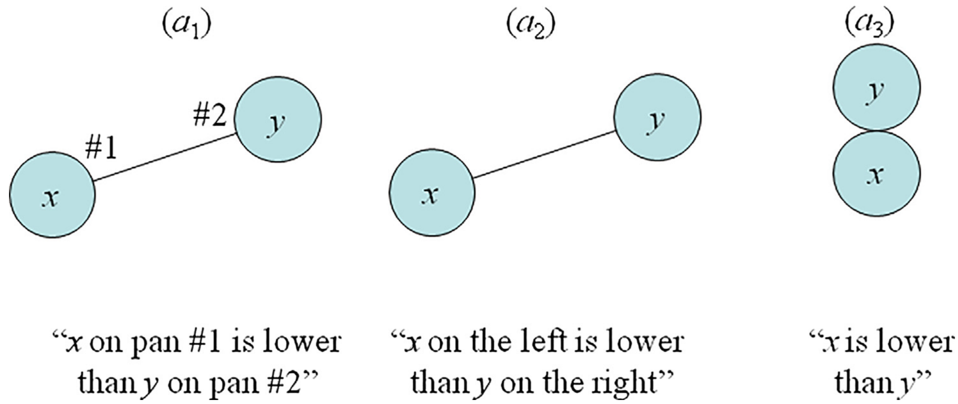

Fig. 1. Three ways to describe the observation of the behavior of a biased balance.

prove the existence of a unique factor that measures the irregularities created by the bias of the balance. We explain that this number cannot be simply interpreted as "the bias of the balance" because it depends on the observer. In Section 5, we assume that we only observe which object is "up" and which object is "down". We again show how we can construct a ratio-scale measuring the mass of objects and a unique factor which can univocally be attributed the balance as measuring its bias. However, this model reflects some indeterminacy in observable behavior. We explain that this indeterminacy stems from the fact that the interaction between the objects and the biased balance is not directly observable. A short Section 6 concludes.

2. Preliminaries

In an *experiment*, the behavior of a biased balance is determined by three types of considerations. First, there are the *objects* which are positioned on its *pans*. Second, there is the measuring device, the *balance* itself, which has a specific *bias,* i.e. whose fulcrum is located at a given place between the two pans. Third, there is the allocation of objects to the pans. Indeed, given two objects and given one biased balance, the behavior of the measuring device is not determined: it depends on the respective positioning of the objects on the pans. This is *the interaction between the objects and the balance*.

We refer to *observation* as the visual perception of the behavior of the balance by an *observer* when objects are positioned on the pans. We describe observations by *pictures* or by *observational statements*. Fig. 1 shows three different ways to observe a particular behavior of a biased balance, with object x placed on one pan and object $y$placed on the other pan.

Each picture can be described by the following observational statements:

(a_1)	x *on pan # 1 is lower than* y *on pan #2*;
(a_2)	x *on the left is lower than* y *on the right*;
(a_3)	x *is lower than* y.

The first statement (a_1) specifies on which pan each object is placed and designates the pans in a manner independent of the observer. Statement (a_2) specifies on which pan each object is positioned in a manner which depends on the observer (left and right are relative to the observer). Statement (a_3) only describes which object tilts the balance, if any. We see below that these three observations lead to three different models.

Let us now consider the *formalization* of observations. We consider a set of objects A and, as we have already done, we denote objects by $x,y,z... \in A$. We designate by $x \circ y$ the object consisting of two objects x and y and we assume that the operation $\circ$ is *closed* (for all $x,y \in A, x \circ y \in A$), *commutative* (for all $x,y \in A, x \circ y = y \circ x$) and *associative* (for all $x,y,z \in A, x \circ (y \circ z) = (x \circ y) \circ z$). The set A endowed with such an operation is a *commutative semigroup*. For $m \in \mathbb{N}^*$, we define mx by $1x = x$ and $mx = (m-1)x \circ x$ where $\mathbb{N}^*$ stands for the set of positive integers. Naturally, mx designs the object consisting of m copies of x (note this already departs from the intuitive analysis where every object is distinct). Further, we assume that A is *homogeneous*, i.e. that given two objects x and y, there exist two positive integers m and n such that $mx = ny$ (note that this hypothesis implies that there is no object of null mass). As we did in the introduction, we formalize the behavior of the balance with a binary relation noted $\succ$ when the balance is not at equilibrium and $\sim$ when the balance is at equilibrium. To characterize the observed regularities of each model and to prove our representation theorems, we use different properties for binary relations on a commutative semigroup. Consider a binary relation R on a commutative semigroup X, we use the following definitions:

- R is *asymmetric* if and only if, for all $x, y \in X$, $xRy \Rightarrow not(yRx)$;
- R is *symmetric* if and only if, for all $x,y \in X$, $xRy \Rightarrow yRx$;
- R is *complete* if and only if, for all $x,y \in X$, xRy *or* yRx;
- R is *transitive* if and only if, for all $x,y,z \in X$, $(xRy$ *and* $yRz) \Rightarrow xRz$;
- R is *positive* if and only if, for all $x,y,z \in X$, $xRy \Rightarrow (x \circ z)Ry$;
- R is *non-trivial* if and only if, for some $x,y,z,t \in X$, xRy *and* $not(zRt)$;
- R is *homothetic* if and only if, for all $x,y \in X$ and all $m \in \mathbb{N}^*$, $xRy \Leftrightarrow mxRmy$;
- R is *additively independent*[2] if and only if, for all $x,y,z \in X$, $xRy \Leftrightarrow (x \circ z)R(y \circ z)$;
- R is *super-archimedean*[3] if and only if, for all $x,y \in X$, $xRy \Rightarrow mxRny$ *for some* $m < n$, with $m,n \in \mathbb{N}^*$.

We also use the following definition, which characterizes the pairs for which the relation R is not super-archimedean[4]:

- A pair $(x,y) \in X \times X$ is *balanced* if and only if xRy *and not*$(mxRny)$ *for all* $m < n$, with $m,n \in \mathbb{N}^*$.

Formalizing the biased balance aims at clarifying the conditions under which the existence and uniqueness of a function that *measures* objects, i.e. which assigns a numerical value (e.g. their mass value) to each of them, can be proved. Such a function is denoted by φ and takes its values in the set of positive real numbers,[5] denoted $\mathbb{R}_{>0}$. According to the theory of representational measurement, this process of representing a relation which formalizes an observed behavior by a

[2] This standard property is also often called *monotonicity*.

[3] This is the term used by De Miguel et al. [7] and that we adopted in Le Menestrel and Lemaire [17].

[4] This property is inspired from the property of *anomalous pairs* in Fuchs [10].

[5] This rules out zero mass objects. Note that this condition is mathematically imposed by the assumption that the set of objects is homogeneous.

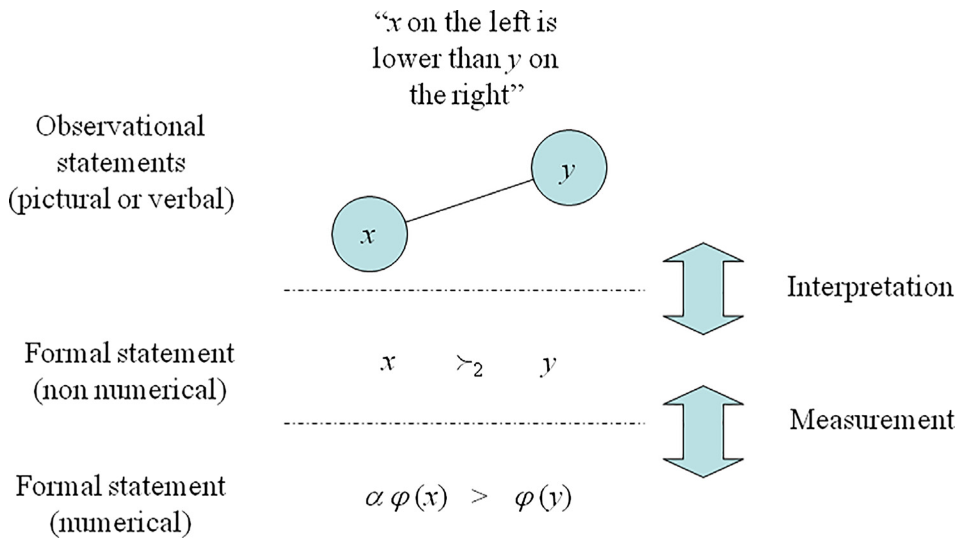

Fig. 2. Interpretation and measurement.

relation among the numbers assigned to the objects is the one of *measurement*. Another objective is to clarify the extent to which we can also provide a quantitative characterization of the measuring device itself, i.e. the bias of the balance. To this purpose, our representations will reveal a *factor*, denoted α or β and belonging to the set of positive real numbers. What such a factor means is indeed a matter of interpretation whenever the balance or its bias is not itself an observable primitive of the model.

By *interpretation*, we refer to the correspondence between something pertaining to an experimental setting with some formal symbol or statement, and reciprocally. This comprises the formalization of a particular observation, or the empirical meaning of a formal symbol or statement. This distinction between interpretation and measurement is depicted in Fig. 2, with the example of statement (a_2), anticipating our second model below.

3. First model: Extending the definition of objects to include the measuring device

In this first model, we consider an observer who can identify each pan of the balance, calling them #1 and #2. Hence, the distinction between the two pans of the balance is independent of the observer (contrary to model 2 below) and is observable (contrary to model 3 below). The principle of this model resides in extending the definition of objects in order to include their interaction with the measuring device. We consider the pair composed of object x and of pan #1 as being an *extended object* and we denote it as $(x,1)$. The possible observations of an experiment are pictured in Fig. 3. We may observe that *x on pan #1 is lower than y on pan #2*, which is written $(x,1) \succ_1 (y,2)$. We may observe that *y on pan #2 is lower than x on pan #1*, which is written $(y,2) \succ_1 (x,1)$. Finally, we observe that *the balance is at equilibrium when x is on pan #1 and y on pan #2*, written $(x,1) \sim_1 (y,2)$, if and only if none of the two previous outcomes are observed.

In order to reflect observation properly, we cannot simply define the relations $\succ_1$ on the set $A \times \{1,2\}$ because we want to reflect formally that, for instance, $(x,1) \succ_1 (y,1)$ is neither true or false but simply not observable since x and y are not on two different pans. We therefore need a slightly modified definition for the relation $\succ_1$.

Let $B = (A_1 \times A_2) \cup (A_2 \times A_1)$ where $A_i = A \times \{i\}$ with $i = \{1,2\}$. We define $\succ_1$ as a subset of B and we say that $\succ_1$ is *restrained*[6] to B. We can define the equilibrium relation $\sim_1$ from the relation $\succ_1$ as: for all $(x,y) \in B$, for all $i \in \{1,2\}, (x,i) \sim_1 (y,3-i) \Leftrightarrow ((x,i) \neg \succ_1 (y,3-i)$ *and* $(y,3-i) \neg \succ_1 (x,i))$. The relation $\succsim_1$ designates the union of the relations $\succ_1$ and $\sim_1$ and is defined as, for all $x,y \in A$ and all $i \in \{1,2\}$: $(x,i) \succsim_1 (y,3-i) \Leftrightarrow ((x,i) \succ_1 (y,3-i)$ *or* $(x,i) \sim_1 (y,3-i))$.

The observable natural regularities of this biased balance are reflected assuming that the relation $\succ_1$ is positive, reflecting that mass is a positive quantity. As in the introduction, the operation combining objects x and y as $x \circ y$ naturally means the operation of jointly positioning two objects on the same pan of the balance. Also, both relations $\succ_1$ and $\sim_1$ are homothetic: the behavior will not change if we take m copies of x and of y ($m \in \mathbb{N}^*$).

Further, the relation $\succ_1$ is super-archimedean but $\sim_1$ is not. Indeed, if the balance is at equilibrium, it will tilt as soon as the ratio of the number of copies is modified. Note that in the theorem below, we use this distinctive property to start from $\succsim_1$ and define the two relations $\succ_1$ and $\sim_1$ from this primitive. Finally, we assume that the relation $\succsim_1$ is non-trivial, i.e. that there exist at least two objects such that the balance tilts. Furthermore, the relation $\succ_1$ is asymmetric ($\sim_1$ is symmetricby construction).[7] Finally, remark that these relations are not additively independent whenever there is a bias.

This model leads to a form of conjoint representation:

Theorem 1. *Let A be a commutative semigroup. Let $A_i = A \times \{i\}$ ($i = \{1,2\}$) and $B = (A_1 \times A_2) \cup (A_2 \times A_1) \subset A \times \{1,2\}$. Let $\succ_1$ be a non-trivial binary relation restrained to B that is asymmetric, positive, homothetic and super-archimedean. Suppose that A is homogeneous. Then there exist a function $\varphi: A \to \mathbb{R}_{>0}$ and two numbers $\beta_1,\beta_2 \in \mathbb{R}_{>0}$ such that, for all $x,y \in A$, all $i \in \{1,2\}$, we have*

$$(x,i) \succ_1 (y,3-i) \Leftrightarrow \beta_i\varphi(x) > \beta_{3-i}\varphi(y), \tag{i}$$

$$\varphi(x \circ y) = \varphi(x) + \varphi(y). \tag{ii}$$

$$\beta_1 + \beta_2 = 1. \tag{iii}$$

Moreover, the pair (β_1,β_2) is unique and the function φ is unique up to multiplication by a positive constant.

Proof. Define the binary relation $\sim_1$ restrained on B by: for all $(x,y) \in B$, for all $i \in \{1,2\}, (x,i) \sim_1 (y,3-i) \Leftrightarrow ((x,i) \neg \succ_1 (y,3-i)$ *and* $(y,3-i) \neg \succ_1 (x,i))$. Define also the relation $\succsim_1$ restrained on B as: for all $(x,y) \in B$, for all $i \in \{1,2\}$: $(x,i) \succsim_1 (y,3-i) \Leftrightarrow ((x,i) \succ_1 (y,3-i)$ *or* $(x,i) \sim_1 (y,3-i))$. Now, from $\succ_1$, we define two binary relations $\succ_1^1$ and

[6] Remark that $B \subset (A \times \{1,2\})^2$ hence $\succ_1$ is *not* a relation defined on $A \times \{1,2\}$.

[7] Because the relation is restrained, the formulation of the transitivity property does not make much sense at this stage.

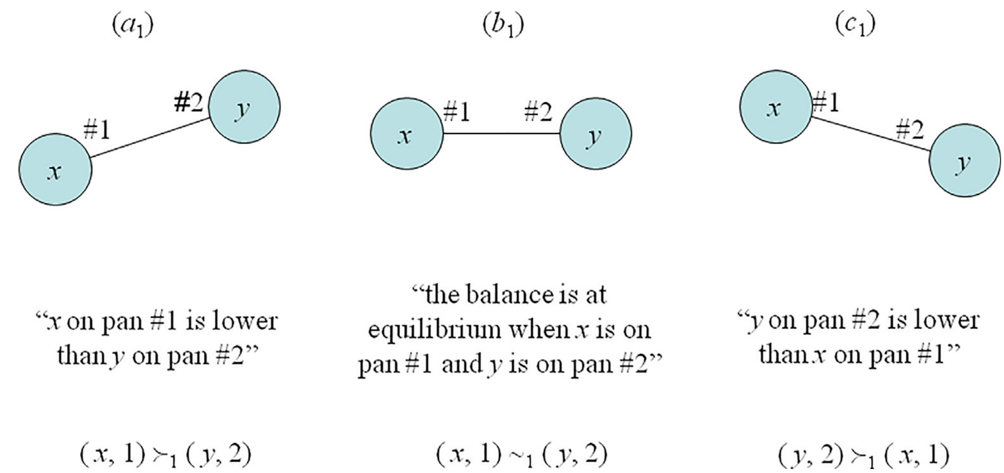

Fig. 3. Observations of Model 1: Extension of the objects with each pan.

$\succ_1^2$ on A:

$x \succ_1^i y \Leftrightarrow (x,i) \succ_1 (y,3-i)$.

Both relations verify conditions of Theorem 1 in Le Menestrel and Lemaire [17]. The additive property (*ii*) is easily obtained from the homogeneity property. To see this, take $x,y \in A$. There exist m,n such that $mx = ny$. Hence, we have $\varphi(x\circ y) = \frac{1}{n}\varphi(nx\circ ny) = \frac{1}{n}\varphi(nx\circ mx) = \frac{n+m}{n}\varphi(x) = \varphi(x) + \frac{1}{n}\varphi(mx) = \varphi(x) + \varphi(y)$.

Hence, for $i = 1,2$, there exists a ratio-scale φ_i: $A \to \mathbb{R}_{>0}$ and a constant $\beta_i \in \mathbb{R}_{>0}$ such that

$x \succ_1^i y \Leftrightarrow \beta_i\varphi_i(x) > (1-\beta_i)\varphi_i(y)$.

Because each φ_i is unique up to multiplication by a positive scalar, we can suppose that $\varphi_1 = \varphi_2 = \varphi$. Moreover, for $i = 1,2$, define the binary relation $\succsim_1^i$ on A by

$x \succsim_1^i y \Leftrightarrow (x,i) \succsim_1 (y,3-i)$.

We have

$x \succsim_1^i y \Leftrightarrow y \neg \succ_1^{3-i} x$.

Since $x \succ_1^i y \Rightarrow x \succsim_1^i y$, we have

$\beta_i\varphi(x) > (1-\beta_i)\varphi(y) \Rightarrow (1-\beta_{3-i})\varphi(x) \geqslant \beta_{3-i}\varphi(y)$,

which is only possible if $\frac{1-\beta_1}{\beta_1} = \frac{\beta_2}{1-\beta_2}$, i.e. if $\beta_2 = 1-\beta_1$. The uniqueness conditions are clear. □

The function φ is naturally interpreted as measuring the mass of the objects. It is additive with respect to the composition of objects even though the additive independence property has not been assumed to describe empirical observation.[8] Naturally, the numbers β_1 and β_2 are interpreted as measuring the distance between the fulcrum and pans #1 and #2 respectively. In this model, these numbers are attributed to a variable that is part of the primitives (namely each of the two pans). We have here an instance of conjoint measurement, which is indeed similar to the distributive triples in Luce and Narens [19]. To make the standard nature of the representation even more explicit, we define the function Φ: $A \times \{1,2\} \to \mathbb{R}_{>0}$ as

$\Phi(x,i) = \beta_i\varphi(x)$.

Then, the binary relation $\succsim_1$ restrained on B can be uniquely extended to a relation $\succsim_1$ that is complete and transitive (i.e. a weak order) on $A \times \{1,2\}$. For all $x,y \in A$ and all $i,j \in \{1,2\}$, we let

$(x,i) \succsim_1 (y,j) \Leftrightarrow \Phi(x,i) \geqslant \Phi(y,j)$.

In this manner, it is possible to obtain a standard representation without a bias but with a two-attribute function: one attribute for the object and one for its extension, i.e. the pan of the balance on which it is placed. In this manner, when the interaction between the objects and the balance can be part of the definition of objects, then the representation of the biased balance is not really biased.

4. Second model: A dependence on the observer

Consider a biased balance placed in front of the observer. Take two objects x and y that are positioned on the left and right pan respectively. As pictured in Fig. 4, we may observe (a_2), i.e. that *x on the left pan is lower than y on the right pan*. This is formally written $x \succ_2 y$. We may observe (b_2), i.e. that *the balance is at equilibrium when x is on the left pan and y on the right pan*. This is written $x \sim_2 y$. Both $\succ_2$ and $\sim_2$ are binary relations defined on the set of objects A.If the observer observes neither of these two, then (c_2) must be observed, i.e. that *y on the right pan is lower than x on the left pan*. If $\succsim_2$ is defined as $x \succsim_2 y \Leftrightarrow (x \succ_2 y$ *or* $x \sim_2 y)$, it formalizes that *x on the left pan is lower when y is on the right pan or the balance is at equilibrium*. Therefore, we can simply formalize that *y on the right is lower than x on the lft* as $x \neg \succsim_2 y$.

The observable natural regularities of this biased balance are reflected assuming that the relation $\succ_2$ is positive, that both relations $\succ_2$ and $\sim_2$ are homothetic and that the relation $\succ_2$ is super-archimedean but $\sim_2$ is not. Finally, we assume that the relation $\succsim_2$ is non-trivial.

What is especially interesting is that the relation $\succ_2$ is not necessarily asymmetric. We may have ($x \succ_2 y$ *and* $y \succ_2 x$) when, for instance, x and y have the same mass and the left arm is longer. As for the relation $\sim_2$, it is not necessarily symmetric: we may have ($x \sim_2 y$ *and* $y \neg \sim_2 x$) whenever arms have different lengths. Also, the relations $\succ_2$ and $\sim_2$ are not necessarily transitive: when the left arm is longer, we may have ($x \succ_2 y, y \succ_2 z$ *and* $x \neg \succ_2 z$) and also ($x \sim_2 y, y \sim_2 z$ *and* $x \neg \sim_2 z$). Further, the relation $\succsim_2$ is not necessarily complete: we may have ($x \neg \succsim_2 y$ *and* $y \neg \succsim_2 x$). This happens, for instance, if x and y have the same mass and the right arm is longer. Finally, the relation $\succsim_2$ is not necessarily additively independent because of a possible lever effect.

In terms of the formal properties of the primitive relations $\succ_2$ and $\sim_2$, this model is thus very general. Despite this generality, we can prove the existence and uniqueness of a numerical function that measures the mass of the objects. We can also provide some sort of measurement of the bias of the balance. This is shown in the following representation theorem:

Theorem 2. *Let A be a commutative semigroup endowed with a nontrivial binary relation $\succsim_2$ that is positive and homothetic. Write $x \sim_2 y$ if and only if (x,y) is balanced and $x \succ_2 y$ if and only if ($x \succsim_2 y$ and $x \neg \sim_2 y$). Suppose*

[8] This is possible because of the both the homotheticity condition and the biased representation.

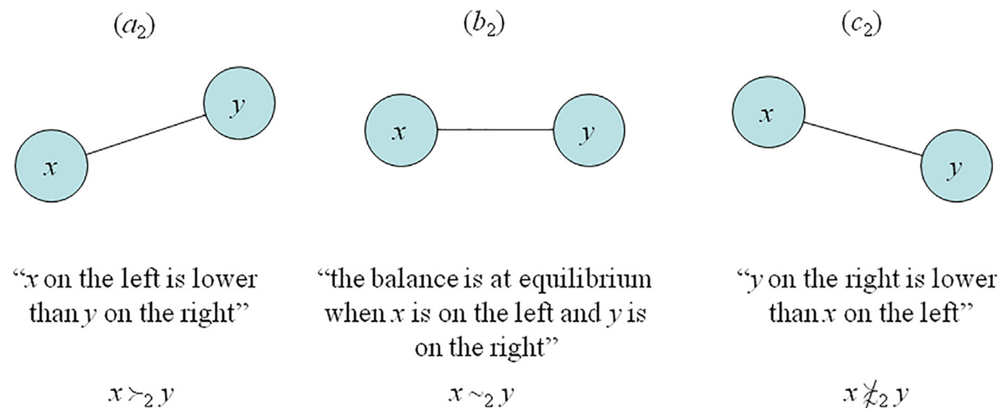

Fig. 4. Observations of Model 2: Left and right dependence on the observer.

A is homogeneous. Then there exist a function $\varphi: A \to \mathbb{R}_{>0}$ *and a number* $\alpha > 0$ *such that we have*

$$x \succ_2 y \Leftrightarrow \alpha\varphi(x) > \varphi(y),$$

$$x \sim_2 y \Leftrightarrow \alpha\varphi(x) = \varphi(y),$$

$$\varphi(x \circ y) = \varphi(x) + \varphi(y).$$

Moreover, φ *is unique up to multiplication by a positive constant and* α *is unique.*

Proof. Suppose there is no balanced pair, then the theorem above amounts to Theorem 1 in Le Menestrel and Lemaire [17]. If there are balanced pairs, then $\succsim_1$ is not super-archimedean and we can use Theorem 2 in Le Menestrel and Lemaire [17].□

A simple corollary of this theorem implies that, in a homogeneous setting, any positive and homothetic relation is either asymmetric and transitive, or it is complete. It is transitive and complete (and then not asymmetric) if and only if the balance is not biased. Also, note that if there are no balanced pairs, no equilibrium can be observed. In that case, the relation $\sim_2$ is empty and we have $\succsim_2 = \succ_2$. In this homogeneous setting, the factor α is then necessarily an irrational number[9].

As in Model 1, the function φ is naturally interpreted as measuring the mass of the objects. Contrary to Mari [20], it is not because the primitive relation is intransitive that numerical measurement is necessarily impossible. Indeed, Theorem 1 shows that measurement of the ratio of two masses is possible, in line with the definition of measurement given by Michell [22, p. 287].

For us, the observer, if the number α measures **the distance between the fulcrum and the left pan**, then **the distance between the fulcrum and the right pan is** 1. These lengths are unique up to multiplication by a positive constant: they constitute a ratio-scale.

The factor α cannot be directly interpreted as measuring “the bias of the balance” because the same experiment (i.e. same objects, same balance, same interaction) may lead to another formalization. Consider another observer placed on the other side of the balance. She would follow the same instructions to formalize her observations. However, when one observer observes outcome (a_2) and formally write $x \succ_2 y$, she would observe outcome (c_2) and write $y \neg \succsim_2' x$, where the $'$ denotes the relation observed by the other observer. In her representation, she would obtain a factor $\alpha' = \frac{1}{\alpha}$ that she may wrongly interpret as measuring “the bias of the balance”. Hence, because “the bias of the balance” may take two distinct values, the numerical factor α shall rather be interpreted as reflecting “the bias of the balance from the point of view of the observer”.

Note that such a dependence can be avoided with another formalization which has less generality. Suppose that we observe that x on the left is lower than y on the right *and* that x on the right is lower than y on the left. We then formalize that an object tilts the balance independently of the pan on which it is positioned. We write $x \gg_2 y$ if and only if ($x \succ_2 y$ *and* $y \neg \succsim_2 x$). The relation $\gg_2$ does not have the same properties than the relation $\succ_2$: it is necessarily asymmetric and transitive. Hence, the factor α is necessarily lower or equal to 1 and the corresponding representation, with less generality, is $x \gg_2 y \Leftrightarrow \alpha\varphi(x) > \varphi(y)$ with $0 < \alpha \leqslant 1$. An example of interpretation of $\gg_2$ distinct from the biased balance consists in formalizing the weighing of objects by hand. One tends to permute objects in the hands in order to get rid of a possible bias when assessing that an object has a greater mass than the other. The procedure leaves out objects whose masses are close, and this lack of discrimination results in a form of “intransitive indifference” with a proportional threshold of just noticeable difference referred to as Weber’s law (See [15] who refer to this interpretation. See also the models in [16] and [14]).

5. Third model: A partial indeterminacy in observable behavior

We consider next a balance placed parallel to our axis of vision. The observer sees which object is lower, if any, but does not observe on which pan each object is positioned. When object x is placed on one pan of the balance and object y on the other pan, we may observe (a_3), i.e. that *x is lower than y*, that we write as $x \succ_3 y$ (Fig. 5).We may observe (b_3), i.e. that *the balance is at equilibrium*, that we formalize as $x \sim_3 y$. Finally, we may observe (c_3), i.e. that *y is lower than x* which we write $y \succ_3 x$. Compared with the previous model, the distinction left and right does not apply. Because we cannot observe on which pan each object is placed, it is not possible to control the permutation of two objects when preparing an experiment.

Suppose that we observe that object x is lower than object y. Of course, it does not mean that the mass of x is greater than the mass of y. Suppose now that we make a second observation, and that this time, with the same balance and the same objects, we observe that x and y are at equilibrium. Then, we can infer that x indeed has a greater mass than y, that x was positioned on the longer pan in the first experiment, and that x was positioned on the shorter pan in the second experiment. From two distinct observations, one being an equilibrium, we just showed how to acquire some information about the interaction between the objects and the measuring device, even though such interaction is not directly observable. Two such observations allow to make deterministic predictions. For instance, the combination of x and another object is necessarily lower than y. Note also that if we had observed y lower than x in the second experiment, we could only have inferred that

[9] Reciprocally, we show in Le Menestrel and Lemaire [17] that if α is irrational, then no equilibrium exists. Note also that in the absence of equilibrium, i.e. if and only if α is irrational, we cannot be certain with a finite number of observations that two objects have the same mass in a homogeneous setting. Hence, the assumption that we dispose of identical copies of an object x becomes especially important.

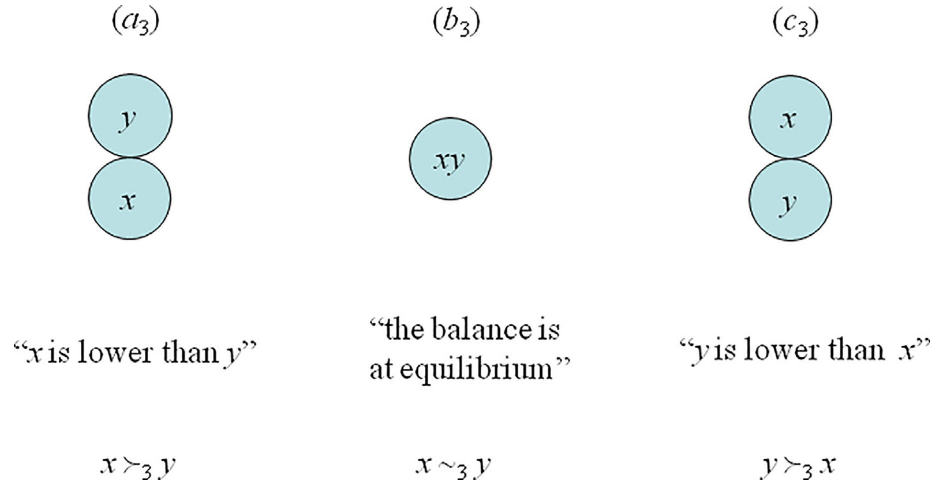

Fig. 5. Observations of Model 3: Partial indeterminacy due to unobservable interaction with the measuring device.

x and y have masses that do not differ "too much", without gaining information about the interaction with the measuring device nor being able to make deterministic predictions.

Formally, the relation $\succsim_3$ designates the union of the relations $\succ_3$ and $\sim_3$ and can again be defined as $x \succsim_3 y \Leftrightarrow (x \succ_3 y \textit{ or } x \sim_3 y)$. As in model 1 and 2, the relation $\succ_3$ is positive and super-archimedean, and the relations $\succ_3$ and $\sim_3$ are homothetic. The relation $\succ_3$ is not necessarily asymmetric and neither $\succ_3$ nor $\sim_3$ are necessarily transitive. On the other hand, the relation $\sim_3$ is symmetric, reflecting that we cannot distinguish between objects at equilibrium. Also, the relation $\succsim_3$ is complete (either one object tilts the balance, or the balance is at equilibrium).

Even if it does not depend on the observer, this model has less mathematical generality than model 1. Its representation, which derives from the representation of the first model, reveals a factor greater or equal to 1:

Theorem 3. *Let A be a commutative semigroup endowed with a nontrivial binary relation $\succsim_3$ that is positive, homothetic and complete. Write $x \sim_3 y$ if and only if (x,y) is balanced or (y,x) is balanced, and $x \succ_3 y$ if and only if ($x \succsim_3 y$ and (x,y) is not balanced). Suppose A is homogeneous. Then there exist a function φ: $A \to \mathbb{R}_{>0}$ and a number $\alpha \geqslant 1$ such that we have*

$$x \succ_3 y \Leftrightarrow \alpha\varphi(x) > \varphi(y),$$

$$x \sim_3 y \Leftrightarrow \begin{cases} \alpha\varphi(x) = \varphi(y) \\ or \\ \varphi(x) = \alpha\varphi(y) \end{cases},$$

$$\varphi(x \circ y) = \varphi(x) + \varphi(y).$$

Moreover, φ is unique up to multiplication by a positive constant and α is unique.

Proof. This is a corollary of Theorem 1. □

Note that the symmetry of the relation $\sim_3$ stems from its symmetrical definition which differs from the one in model 2. This is because x and y may put the balance at equilibrium while the pair (y,x) is not balanced. We have the peculiar property that the relations $\succ_3$ and $\sim_3$ are not disjoint, i.e. we may have $x \succ_3 y$ *and* $x \sim_3 y$. Because of this, it is not possible in this model to know *with certainty* that the balance is not biased.[10]

The function φ has the same interpretation as in model 2. For the observer, if the number α is interpreted as measuring **the longer arm**, then **the shorter arm**is of length 1. These lengths constitute a ratio-scale. Of course, since we cannot observe the arms, it is not possible to say which one is the longer arm or the shorter.

Apart from the cases where deterministic predictions can be made, one does not know which outcome is going to be observed in this model. This partial indeterminacy is not due to insufficient knowledge about the objects, nor to insufficient knowledge about the measuring device, but to a lack of knowledge about *the interaction between the objects and the measuring device*. What appears, from the point of view of the observer, as the same "observable cause" (two given objects on a given measuring device), does not lead to the "same observable effect." This illustrates a violation of *procedural invariance* where two substantially equivalent settings are not empirically equivalent (see e.g. [26]). In other words, observation depends on how the measuring device treats the objects.[11] Note that the three statements $x \succ_3 y$,$x \sim_3 y$ and $y \succ_3 x$ which describe the possible observations in this model are not mutually exclusive. Hence, the correspondence between observational statements and formal statements is not one-to-one. A formal statement may have two distinct meaning in terms of observational statements. For instance, if the observational statement ***x** is lower than **y*** necessarily implies the formal statement $x \succ_3 y$, the formal statement $x \succ_3 y$ does not imply that we will necessarily observe that ***x** is lower than **y***. It merely means that observing that ***x** is lower than **y*** is *possible*. It is as if there were a time asymmetry in the sense that a formal statement can correspond either to a statement of fact (a description of an observation that took place in the past) either to a statement of possibility (a prediction of an observation that may take place in the future).[12] The relation $\succsim_3$ can be easily obtained from the relation $\succsim_2$ of the second model. Observing that x is lower than y means that either x on the left is lower than y on the right, or x on the right is lower than y on the left. Formally, we have $x \succsim_3 y \Leftrightarrow (x \succsim_2 y \textit{ or } y \neg \succsim_2 x)$. More precisely, the relation $\succsim_3$ is formally the converse of the negation of the relation $\gg_2$: we have $x \succsim_3 y \Leftrightarrow y \neg \gg_2 x$. Note that the relation $\gg_2$ itself is *not directly observable in this model*. However, it can be derived from observation in some particular cases. For instance, we may observe ($x \succ_3 y$ *and* $y \sim_3 x$) and deduce that $x \circ z \gg_2 y$ for all $z \in A$. In this manner, the relation $\gg_2$ characterizes the pairs for which a deterministic prediction can be made.

[10] Here, our intuition suggests that observing an equilibrium between x and y in a large number of experiments probably means that the balance is not biased, and that x and y have the same mass. But this necessitates a probabilistic approach, with an hypothesis about the random generating process assigning objects to the pans. This would be an interesting avenue for further research with links with probabilistic models.

[11] This is specially interesting for the modeling of preferences, which are often dependant on the process by which they are constructed (e.g. [30,12], and also [29,18]).

[12] Somewhat similar statements have been made in the context of quantum mechanics (e.g. [1]).

6. Conclusion

This paper provides a rigorous analysis of experiments where objects and a dissymmetric measuring device combine to produce an observable phenomenon. Under idealized and abstract conditions (uniform gravitational field, no uncertainty, no influence quantities, etc.), it illustrates how measurement is possible even when the observed relation between objects is incomplete, intransitive or does not verify additive independence. It clarifies minimal conditions under which it is possible to treat the interaction with the measuring device as one dimension of some "extended objects" (Model 1). It also shows how interpretation of observation may lead to a formalism that is dependent on the observer (Model 2) or to partial indeterminacy in observable behavior (Model 3). Finally, The resulting form of "biased measurement" illustrated by the biased balance extends measurement to relational structures that cannot be represented by a function only, but can be represented by a function and a bias. The biased balance also allows to study the relation between measurement and empirical observation in more details. It illustrates how a biased measuring device which is neither directly observable nor a primitive influences the observed relation between objects. Thereby, it shows some extent to which a measuring process device can be itself quantified.

For these models, the bias of the balance has been assumed to be constant. Although this is a natural assumption to uncover the mathematical and methodological peculiarities of such a device, it would be interesting to study a biased balance whose fulcrum moves according to specific properties. Also in terms of limitations, these insights are derived from a model of measurement based on direct comparison. It would be most interesting to study how this relates with the calibration of the measuring instrument when the measurement is based on calibrated sensors. Furthermore, it would be interesting to assess whether the homothetic property -which substitute for the additive independence property, is also a structural property in other significant cases of metrology. Methodologically, the biased balance nevertheless provides a concrete grasp to the distinctive roles played by the observer, the measuring device and the objects, notions which usually pertain to the theory of quantum mechanics (e.g. [35]) or to the philosophy of science (e.g. [33]).

For the sake of simplicity, this paper has been written under the homogeneity assumption for the objects under measurement. With Bertrand Lemaire, we have generalized the theory of biased measurement in a non-homogeneous setting [14]. These theorems allow for biases that are not constant but do not fundamentally change the methodological approach presented here. Notwithstanding, a generalization of Model 1 for the case of a balance with n pans will certainly be an interesting generalization of the notion of a Grassman structure (see [34, p. 229]).

Finally, the biased balance is a powerful model for the study of biased judgments. Human beings cannot be viewed as deities who judge things without biases, an assumption which is however the cornerstone of rational behavior as maximization of a utility function. Human judgements involve both some observable objects and the subject himself, who is looking at the objects with his specific values, in a manner that can be specific to the situation at hand. It is as if the subject constructs his preferences by positioning himself towards the objects he judges. These models of the biased balance should help to better reflect how such attitudes influence preferences. At last, it could also provide a measurement theoretic approach for studies that combine observation of behavior with observation of the human brain [6].

References

[1] M. Bitbol, The concept of measurement and time symmetry in quantum mechanics, Philosophy Sci. 55 (1988) 349–375.
[2] M. Boumans, Measurement in economic systems, Measurement 38 (2005) 275–284.
[3] N.R. Campbell, Foundations of Science, Dover Publications, New York, 1957.
[4] R. Carnap, An Introduction to the Philosophy of Science, Dover Publications, New York, 1966–1995.
[5] J. Chevalier, A. Gheerbrant, Dictionnaire des Symboles, Robert Laffont, Paris, 1969.
[6] B. De Martino, D. Kumaran, B. Seymour, R.J. Dolan, Frames, biases, and rational decision-making in the human brain, Science 313 (2006) 684–687.
[7] J.R. De Miguel, J.C. Candeal, E. Indurain, Archimedeaness and additive utility on totally ordered semigroups, Semigroup. Forum. 52 (1996) 335–347.
[8] B. Ellis, Basic Concepts of Measurement, Cambridge University Press, Cambridge, 1966.
[9] D.H. Krantz, R.D. Luce, P. Suppes, A. Tversky, Foundations of measurement, Additive and Polynomial Representations vol. I, Academic Press, San Diego, 1971.
[10] L. Fuchs, Partially Ordered Algebraic Systems, Pergamon Press, 1963.
[11] T.L. Heath (Ed.), The Works of Archimedes with the Method of Archimedes, Dover, New York, 1897.
[12] D. Kahneman, A. Tversky, Choices, Values and Frames, Cambridge University Press, New York, 2000.
[13] B. Kish, Scales and Weights: A Historical Outline, Yale University Press, New Haven and London, 1965.
[14] B. Lemaire, M. Le Menestrel, Homothetic interval orders, Discrete Mathe. 306 (2006) 1669–1683.
[15] M. Le Menestrel, B. Lemaire, Biased extensive measurement: the homogeneous case, J. Math. Psychol. 48 (2004) 9–14.
[16] M. Le Menestrel, B. Lemaire, Biased extensive measurement: the general case, J. Math. Psychol. 50 (2006) 570–581.
[17] M. Le Menestrel, B. Lemaire, Ratio-scale measurement with intransitivity or incompleteness: the homogeneous case, Theor. Decis. 60 (2006) 207–217.
[18] M. Le Menestrel, L. Van Wassenhove, The domain and interpretation of utility functions: an exploration, Theor. Decis. 51 (2001) 329–349.
[19] R.D. Luce, L. Narens, Classification of concatenation structures according to scale type, J. Math. Psychol. 29 (1985) 9–72.
[20] L. Mari, Beyond the representational viewpoint: a new formalization of measurement, Measurement 27 (2000) 71–84.
[21] L. Mari, The problem of foundations of measurement, Measurement 38 (2005) 259–266.
[22] J. Michell, The logic of measurement: a realist overview, Measurement 38 (2005) 285–294.
[23] J. Michell, C. Ernst, The axioms of quantity and the theory of measurement. Translated from Part I of Otto Hölder's German Text "Die Axiome der Quantität and die Lehre vom Mass, J. Math. Psychol. 40 (1996) 235–252.
[24] E.A. Moody, M. Clagett, The Medieval Science of Weights (Scientia de Ponderibus), The university of Wisconsin Press, Madison, 1960.
[25] L. Narens, Abstract Measurement Theory, MIT Press, Cambridge, 1985.
[26] W.V. Quine, On empirically equivalent systems of the world, Erkenntnis 9 (1975) 313–328.
[27] F.S. Roberts, Measurement Theory with Applications to Decision Making, Utility and the Social Sciences, Addison-Wesley, Reading, 1979.
[28] K.W. Schwager, The representational theory of measurement: an assessment, Psychol. Bull. 110 (1991) 618–626.
[29] A. Sen, Information and invariance in normative choice, Social Choice and Public Decision Making, Cambridge University Press, Cambridge, UK, 1986, pp. 29–55.
[30] P. Slovic, The construction of preferences, Am. Psychol. 50 (1995) 271–364.
[31] S. Stevens, On the theory of scales of measurement, Science 103 (1946) 122–136.
[32] P. Suppes, Limitations of the axiomatic method in ancient Greek mathematical sciences, in: J. Hintikka, D. Gruender, E. Agazzi (Eds.), Pisa Conference Proceedings, vol. I, D. Reidel Publishing Company, 1980, pp. 197–213.
[33] P. Suppes, Models and Methods in the Philosophy of Science: Selected Essays, Kluwer Academic Publishers, Dordrecht, 1993.
[34] P. Suppes, D.H. Krantz, R.D. Luce, A. Tversky, Foundations of measurement, Geometrical, Threshold, and Probabilistic Representations vol. II, Academic Press, San Diego, 1989.
[35] J. von Neumann, foundations of Quantum Mechanics, Princeton University Press, 1955.