Generalized homothetic biorders

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Abstract

In this paper, we study the binary relations R on a nonempty \( \mathbb{N}^* \)-set \( A \) which are \( h \)-independent and \( h \)-positive (cf. the Introduction below). They are called homothetic positive orders. Denote by \( B \) the set of intervals of \( \mathbb{R} \) having the form \([r, +\infty)\) with \( 0 < r \leq +\infty \) or \([q, \infty]\) with \( q \in \mathbb{Q}_{>0} \). It is a \( \mathbb{Q}_{>0} \)-set endowed with a binary relation \( \succ \) extending the usual one on \( \mathbb{R}_{>0} \) (identified with a subset of \( B \) via the map \( r \mapsto [r, +\infty) \)). We first prove that there exists a unique map \( \Phi_k : A \times A \to B \) such that (for all \( x, y \in A \) and all \( m, n \in \mathbb{N}^* \)) we have \( \Phi(mx, ny) = mn^{-1} \cdot \Phi(x, y) \) and \( x \succ y \iff \Phi_k(x, y) \succ Q \). Moreover, if we impose some natural conditions on the sets \( u_1(A) \) and \( u_2(A) \), the representation \((u_1, u_2)\) is “uniquely” determined by \( R \). For a generalized homothetic biorder \( R \) on \( A \), the binary relation \( R_1 \) on \( A \) defined by \( x \mathrel{R_1} y \iff \Phi_k(x, y) = \Phi_k(y, x) \) is a generalized homothetic weak order; i.e., there exists a morphism of \( \mathbb{N}^* \)-sets \( u : A \to B \) such that (for all \( x, y \in A \)) we have \( x \mathrel{R_1} y \iff u(x) \succ u(y) \). As we did in [B. Lemaire, M. Le Menestrel, Homothetic interval orders, Discrete Math. 306 (2006) 1669–1683] for homothetic interval orders, we also write “the” representation \((u_1, u_2)\) of \( R \) in terms of \( u \) and a twisting factor.

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This paper proposes a generalization of [13] in which we had studied homothetic interval orders on a nonempty \( \mathbb{N}^* \)-set \( A \). Let us recall that such an order \( R \) is a nonempty binary relation, \( h \)-independent in the sense that \( x \succ y \iff mx \succ ny \) for all \( x, y \in A \) and all \( m, n \in \mathbb{N}^* \), and satisfying a series of properties that ensure the existence of two morphisms of \( \mathbb{N}^* \)-sets \( u_1, u_2 : A \to \mathbb{R}_{>0} \) such that \( x \succ y \iff u_1(x) > u_2(y) \) with \( u_1 \leq u_2 \). Moreover, the pair \((u_1, u_2)\) is unique up to multiplication by a positive scalar. Besides \( h \)-independence, the most striking properties of homothetic interval orders are:

- asymmetry: \( x \succ y \implies y \prec x \),
- \( h \)-positivity: for all \( m, n \in \mathbb{N}^* \) such that \( m > n \), we have \( x \succ y \implies mx \succ ny \),
- \( h \)-super-Archimedean: if \( x \succ y \), then there exists \( m \in \mathbb{N}^* \) such that \( mx \succ (m + 1)y \).

Note that asymmetry implies

- irreflexivity: \( x \not\prec x \).

Of all these properties, this paper first retains only two: \( h \)-independence and \( h \)-positivity.

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1 In [13], we called this property \( h \)-Archimedean but the terminology of the present paper is more in line with the literature (see e.g. [6]).
Let \( \mathcal{R}(A) \) be the set of binary relations on \( A \) that are \( h \)-independent and \( h \)-positive. Abandoning the \( h \)-super-Archimedean property naturally leads to enrich the range of the representation: denote by \( \mathbb{R}^2 \) the set of intervals of \( \mathbb{R} \) having the form \([r, +\infty) \) or \([r, +\infty[ \) with \(-\infty < r \leq +\infty \), endowed with the order \( \leq \) inverse of the one given by inclusion. We identify \( r \in \mathbb{R} \) with the closed interval \([r, +\infty] \), and we note \( r^+ \) the open interval \([r, +\infty[ \). We denote by \( \mathbb{R}^2 \) the empty interval \( ] -\infty, +\infty[ \). In this manner, the relation \( \leq \) on \( \mathbb{R}^2 \) extends the relation on \( \mathbb{R} \), and we denote by \( > \) its negation: for two intervals \( I, I' \) in \( \mathbb{R}^2 \), we then have \( I > I' \iff I' \not\subset I \). We also endow \( \mathbb{R}^2 \) with a \( \mathbb{R}_0^-\) set structure extending the one of \( \mathbb{R} \) (cf. Section 1). Finally, we define:

\[
\mathbb{B} = \mathbb{R}_{>0} \cup \{q^+ : q \in \mathbb{Q}_{>0} \} \cup \{0^+, \infty\},
\]

\[
\mathbb{A} = \mathbb{R}_{>0} \cup \{0^+, \infty\}.
\]

We have the inclusions \( \mathbb{A} \subset \mathbb{B} \subset \mathbb{R}^2 \); and \( \mathbb{A} \) and \( \mathbb{B} \) are respectively a sub-\( \mathbb{R}_{>0} \)-set and a sub-\( \mathbb{Q}_{>0} \)-set of \( \mathbb{R}^2 \).

Our first result is the following (4.2): for all relation \( R \in \mathcal{R}(A) \), there exists a unique function \( \Phi_R : A \times A \rightarrow \mathbb{B} \) satisfying (for all \( x, y \in A \) and all \( m, n \in \mathbb{N}^+ \)):

1. \( \Phi_R(mx, ny) = \frac{m}{n} \cdot \Phi_R(x, y) \);
2. \( xRy \iff \Phi_R(x, y) > 1 \).

Conversely, any binary relation \( R \) on \( A \) such that there exists a function \( \Phi : A \times A \rightarrow \mathbb{B} \) satisfying (1) and (2) belongs to \( \mathcal{R}(A) \).

Denote by \( \mathcal{R}'(A) \) the subset of \( \mathcal{R}(A) \) consisting of \( h \)-super-Archimedean relations. We verify that for \( R \in \mathcal{R}(A) \), we have \( R \in \mathcal{R}'(A) \) if and only if \( \Phi_R(A \times A) \subset \mathbb{A} \).

We then introduce a notion that extends homothetic interval orders: a relation \( R \in \mathcal{R}(A) \) is said to be a generalized homothetic biorder if there exist two morphisms of \( \mathbb{N}^+ \)-sets \( u_1, u_2 : A \rightarrow \mathbb{R}_{>0}^2 \) such that \( xRy \iff u_1(x) > u_2(y) \); in which case we say that the pair \( (u_1, u_2) \) represents \( R \). We show that if \( R \in \mathcal{R}(A) \) is a generalized homothetic biorder, then the pair \( (u_1, u_2) \) which represents \( R \) is unique up to multiplication by a positive scalar \( (6.5) \). In fact, the correct formulation of this uniqueness property is slightly more complicated (cf. Section 6), and leads us to distinguish three cases: \( \Phi_R(A \times A) \subset \{0^+, \infty\} \); \( \Phi_R(A \times A) \subset \mathbb{A} \setminus \{0^+, \infty\} \); and \( \Phi_R(A \times A) \subset \mathbb{B} \setminus \mathbb{A} \). In particular, if \( R \) is an \( h \)-super-Archimedean generalized homothetic biorder on \( A \) such that \( \Phi_R(A \times A) \not\subset \{0^+, \infty\} \), then there exists a “unique” representation \( (u_1, u_2) \) of \( R \) in \( \mathbb{A} \), i.e. such that \( u_i(A) \subset \mathbb{A} \) (\( i = 1, 2 \)).

Before going on, let us consider the three following properties for a binary relation \( R \) on \( A \):

- **Ferrers:** \( xRy \) and \( zRt \Rightarrow xRt \) or \( zRy \);
- **negative transitivity:** \( x(-R)y(-R)z \Rightarrow x(-R)z \).

Recall that \( R \) is called:

- a **biorder** if it is Ferrers;
- an **interval order** if it is irreflexive and Ferrers;
- a **weak order** if it is asymmetric and negatively transitive.

So we have the implications:

\[
\text{weak order} \Rightarrow \text{interval order} \Rightarrow \text{biorder}.
\]

Usually, a biorder \( R \) on \( A \) is said to be **representable** if there exist two functions \( u_1, u_2 : A \rightarrow \mathbb{R}_{>0} \) such that

\[
xRy \iff u_1(x) > u_2(y).
\]

Note that if \( u_1 \leq u_2 \) (resp. \( u_1 = u_2 \)), then \( R \) is an interval order (resp. a weak order).

Thus homothetic interval orders are particular cases of representable interval order. And the notion of generalized homothetic biorder is a twofold extension: firstly we enlarge the space of the representation, in the sense that the two functions \( u_1 \) and \( u_2 \) on \( A \) may take their values in \( \mathbb{R}_{>0}^2 \) instead of \( \mathbb{R}_{>0} \); secondly we remove the condition \( u_1 \leq u_2 \). A generalized homothetic biorder represented by a pair \( (u_1, u_2) \) such that \( u_1 \leq u_2 \) (resp. \( u_1 = u_2 \)) is called a generalized homothetic interval order (resp. a generalized homothetic weak order).

Let us return to the contents of the paper. We then study the characterization of the subset \( \mathcal{R}_h(A) \subset \mathcal{R}(A) \) of generalized homothetic biorders. Note that in [12], we have studied the case where \( A \) is homogeneous (i.e. such that for all \( x, y \in A \), there exist \( m, n \in \mathbb{N}^+ \) such that \( mx = ny \)). In that special case, it is easy to show that any positive homothetic order is also a homothetic biorder, i.e. we have \( \mathcal{R}_h(A) = \mathcal{R}(A) \). But this equality is no longer true in general. In Section 9, we identify a (finite!) set of properties that characterize \( \mathcal{R}_h(A) \). These properties are in fact compatibility properties between \( R \) and its dual relation \( R^\circ \in \mathcal{R}(A) \), defined by

\[
xR^\circ y \iff y(-R)x.
\]

We show that if \( R \in \mathcal{R}_h(A) \), then the binary relation \( R_1 \) on \( A \) given by

\[
xR_1 y \iff \Phi_R(x, y) > \Phi_R(y, x)
\]
It is an involution: for \( u \in \mathbb{R}_{>0} \), we have \( (u^\vee)^\vee = u \). In particular, it is a bijective map. And for \( r, s \in \mathbb{R}_{>0}^\times \), we have
\[
 r \leq s \iff r^\vee \geq s^\vee.
\]
Let \( R_{>0} \times R_{>0}^5 \rightarrow R_{>0}^5 \), \((s, r) \) be the \( R_{>0} \)-set structure on \( R_{>0}^5 \) defined by \((s, r) \in R_{>0} \):

\[
\begin{align*}
s \cdot r &= sr, & s \cdot r^\vee &= (sr)^+, \\
s \cdot 0^+ &= 0^+, & s \cdot \infty &= \infty.
\end{align*}
\]

For \((s, r) \in R_{>0} \times R_{>0}^5 \), we have

\[(s \cdot r)^\vee = s^{-1} \cdot r^\vee.\]

Let

\[
\begin{align*}
B &= R_{>0} \cup \{q^+: q \in Q_{>0} \} \cup \{0^+, \infty\}, \\
A &= R_{>0} \cup \{0^+, \infty\}, \\
Q_{>0}^5 &= Q_{>0} \cup \{q^+: q \in Q_{>0} \} \cup \{0^+, \infty\}.
\end{align*}
\]

We have the inclusions \( A \subseteq B \) and \( Q_{>0}^5 \subseteq B \). Moreover, we have \( R_{>0} \cdot A = A, Q_{>0} \cdot B = B \) and \( Q_{>0} \cdot Q_{>0}^5 = Q_{>0}^5 \); i.e. \( A \) is a sub-\( R_{>0} \)-set of \( R_{>0} \), and \( B \) and \( Q_{>0}^5 \) are sub-\( Q_{>0} \)-sets of \( R_{>0}^5 \). The involution \( x \mapsto x^\vee \) induces by restriction three bijective maps

\[
\begin{align*}
B &\rightarrow B^\vee = \{r^+: r \in R_{>0} \} \cup Q_{>0} \cup \{\infty\}, \\
A &\rightarrow A^\vee = \{r^+: r \in R_{>0} \} \cup \{\infty\}, \\
Q_{>0}^5 &\rightarrow Q_{>0}^5.
\end{align*}
\]

And we have

\[
\begin{align*}
B \cup B^\vee &= A \cup A^\vee = R_{>0}^5, \\
B \cap B^\vee &= A \cap A^\vee = Q_{>0}^5.
\end{align*}
\]

**Remark 1.1.** Let \( r, s \in B \) such that \( r > s \). Then \( r \neq 0^+, s \neq \infty \), and either \( r \neq s^+ \) and then there exists a \( q \in Q_{>0} \) such that \( r > q > s \); or \( r = s^+ \) and then \( s \in Q_{>0} \). So in both cases, there exists a \( q \in Q_{>0} \) such that \( r > q \geq s \). \( \star \)

2. The sets \( R(A), R'(A) \) and \( R''(A) \)

Recall the definition of \( R(A) \) and \( R'(A) \) given in the introduction. Denote by \( R_0 \) and \( R_\infty \) the binary relations on \( A \) that are respectively empty and trivial; i.e. for all \( x, y \in A \), we have \( x(-R_0)y \) and \( x R_\infty y \) both belong to \( R'(A) \). And we have \( R_0^\vee = R_\infty \) and \( R_\infty^\vee = R_0 \).

The set \( R(A) \) is endowed with a structure of a \( Q_{>0} \)-set: for \( R \in R(A) \) and \( q \in Q_{>0} \), we write \( q = \frac{m}{n} \) with \( m, n \in \mathbb{N}^* \), and we denote \( R^m \) the binary relation on \( A \) defined by \( x R^m y \Leftrightarrow mx R ny \). By \((\_\_1)\), the relation \( R^m \) is well-defined, i.e. it does not depend on the choice of \( m \) and \( n \) such that \( q = \frac{m}{n} \). It clearly belongs to \( R(A) \). For \( R \in R(A) \) and \( q, q' \in Q_{>0} \), we have

\[(R^m)^{q'} = R^{mq'}.\]

And the subset \( R'(A) \) of \( R(A) \) is \( Q_{>0} \)-stable.

For \( R \in R(A) \), we denote \( R' \) the binary relation on \( A \) defined by

\[x R' y \Leftrightarrow (\text{for all } m, n \in \mathbb{N}^* \text{ such that } m > n, \text{ we have } mx R ny).\]

Then \( R' \in R'(A) \), and the map \( R(A) \rightarrow R'(A), R \mapsto R' \) is the identity relation on \( R'(A) \).

For \( R \in R'(A) \), the relation \( R' \) still belongs to \( R'(A) \) (easy to check). And the map \( R'(A) \rightarrow R'(A), R \mapsto R' \) is an involution: for \( R \in R'(A) \), we have \( (R')^\vee = R \). For \( R \in R(A) \) and \( q \in Q_{>0} \), we have

\[
\begin{align*}
(R^m)^{q'} &= (R'^m)^{-1}, \\
(R^m)^{q'} &= (R^m)^{q'}.
\end{align*}
\]

Let \( R'(A)^\vee \) be the subset of \( R(A) \) formed by relations satisfying the following condition \((\_\_A)^\vee \) (for all \( x, y \in A \)):

\((\_\_A)^\vee \): If \((m + 1)x R my \) for all \( m \in \mathbb{N}^* \), then \( x R y \).

The involution \( R(A) \rightarrow R(A), R \mapsto R'' \) induces by restriction two bijective maps, that are the inverse of one another:

\[
\begin{align*}
R'(A) &\rightarrow R'(A)^\vee, \\
R'(A)^\vee &\rightarrow R'(A).
\end{align*}
\]

Let \( R''(A) = R'(A) \cap R'(A)^\vee \). The involution \( R(A) \rightarrow R(A), R \mapsto R' \) induces by restriction a bijective map

\[
R''(A) \rightarrow R''(A).
\]

Notice that since \( R_0, R_\infty \in R'(A) \) and \( R_\infty = R_0^\vee \), we have \( R_0 \in R''(A) \).
3. The invariants $\mathcal{P}_{x,y}^R$, $s_{x,y}^R$ and $t_{x,y}^R$

Let $R \in \mathcal{R}(A)$, and let $x, y \in A$. Let
\[
\mathcal{P}_{x,y}^R = \{ m n^{-1} : m, n \in \mathbb{N}^*, m x R n y \} \subset \mathbb{Q}_{>0}.
\]
If $q \in \mathcal{P}_{x,y}^R$, then we have the inclusion $\mathbb{Q}_{\ge q} \subset \mathcal{P}_{x,y}^R$. Therefore
\[
s_{x,y}^R = \bigcup_{q \in \mathcal{P}_{x,y}^R} [q, +\infty[
\]
is an element of $\mathbb{R}^+$, and we have $s_{x,y}^R \cap \mathbb{Q} = \mathcal{P}_{x,y}^R$. We distinguish two cases: either for all $q \in \mathcal{P}_{x,y}^R$, we have $\mathcal{P}_{x,y}^R \cap \mathbb{Q}_{ \ge q} \neq \emptyset$, and then
\[
s_{x,y}^R = \begin{cases}
[\inf(\mathcal{P}_{x,y}^R)]^+ & \text{if } \mathcal{P}_{x,y}^R \neq \emptyset \\
\infty & \text{if not};
\end{cases}
\]
or there exists a $s \in \mathbb{Q}_{>0}$ such that $\mathcal{P}_{x,y}^R = \mathbb{Q}_{\ge s}$, and then $s_{x,y}^R = s$. In particular, we have $s_{x,y}^R \in \mathbb{Q}^\times$. The triplet $(x, R, y)$ is said to be super-Archimedean in the first case, and non super-Archimedean in the second case. Notice that $R \in \mathcal{R}'(A)$ if and only if for all $x', y' \in A$, the triplet $(x', R, y')$ is super-Archimedean.

**Notation 3.1.** For $R \in \mathcal{R}(A)$, we note $\mathcal{A}^R$ the set of $(x, y) \in A \times A$ such that the triplet $(x, R, y)$ is super-Archimedean, and we let $\mathcal{B}^R = (A \times A) \setminus \mathcal{A}^R$. We also define
\[
\mathcal{A}_1^R = \{ x \in A : (x, y) \in \mathcal{A}^R, \forall y \in A \},
\]
\[
\mathcal{A}_2^R = \{ y \in A : (x, y) \in \mathcal{A}^R, \forall x \in A \},
\]
and
\[
\mathcal{B}_i^R = A \setminus \mathcal{A}_i^R \quad (i = 1, 2).
\]

For $R \in \mathcal{R}(A)$, since $\mathcal{B}^R \subset \mathcal{B}_1^R \times \mathcal{B}_2^R$, we have the decomposition
\[
\mathcal{A}^R = \mathcal{A}^R \cap (\mathcal{B}_1^R \times \mathcal{B}_2^R) \bigsqcup \mathcal{A}^R \cap (\mathcal{B}_1^R \times \mathcal{A}_2^R) \bigsqcup \mathcal{A}^R \cap (\mathcal{A}_1^R \times \mathcal{B}_2^R) \bigsqcup \mathcal{A}_1^R \times \mathcal{A}_2^R. \tag{3.2}
\]

Let $R \in \mathcal{R}(A)$, and let $x, y \in A$. Let
\[
t_{x,y}^R = (s_{x,y}^R)^\vee \in \mathbb{B}.
\]
From what precedes, we have

\((x, R, y) \in \mathcal{A}^R \iff t_{x,y}^R \in \mathbb{A}^R\).

Moreover, we have
\[
x R y \iff 1 \in \mathcal{P}_{x,y}^R \iff s_{x,y}^R \leq 1 \iff t_{x,y}^R \geq 1^+ \iff t_{x,y}^R > 1.
\]

And for all $m, n \in \mathbb{N}^*$, we have
\[
t_{m x, n y}^R = \frac{m}{n} \cdot t_{x,y}^R.
\]

**Lemma 3.4.** Let $R \in \mathcal{R}(A)$.

(1) For $(x, y) \in \mathcal{A}^R$, we have
\[
t_{y,x}^{R^\vee} = \begin{cases}
r & \text{if } s_{x,y}^R = r^+ \text{ with } r \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0} \\
s_{x,y}^R & \text{if } s_{x,y}^R \in [0^+, \infty) \text{ or if } s_{x,y}^R = q^+ \text{ with } q \in \mathbb{Q}_{>0}.
\end{cases}
\]

(2) For $(x, y) \in \mathcal{B}^R$, we have
\[
t_{y,x}^{R^\vee} = s_{x,y}^R \in \mathbb{Q}_{>0}.
\]
Proof. Let \((x, y) \in A \times A\). We have \(R^R_{x,y} = \{q^{-1} : q \in Q_{>0} \setminus \mathcal{P}_x^R y\}\).

Suppose first that \(t^R_{x,y} \in [0^+, \infty]\). Then \((y, x) \in A^R\) and
\[
t^R_{x,y} = s^R_{x,y} \in (0^+, \infty),
\]
Suppose now that \(t^R_{x,y} \in \mathbb{R}_{>0}\). We thus have \(s^R_{x,y} = r^+\) for a \(r \in \mathbb{R}_{>0}\), and
\[
\mathcal{P}^R_{x,y} = Q_{>0} \cap [r^{-1}, +\infty[.
\]
We distinguish two cases: either \(r \in \mathbb{R}_{>0} \setminus Q_{>0}\), and then \((y, R^y, x) \in A^{R^y}\) and \(t^R_{y,x} = r\); or \(r \in Q_{>0}\), and then \((y, R^y, x) \in B^{R^y}\) and \(t^R_{y,x} = r^+\).

Suppose finally that \(t^R_{x,y} \in B \setminus A\). We thus have \(s^R_{x,y} = q\) for a \(q \in Q_{>0}\), and
\[
\mathcal{P}^R_{x,y} = Q_{>0} \cap ]q^{-1}, +\infty[.
\]
Therefore, we have \((y, R^y, x) \in A^{R^y}\) and \(t^R_{y,x} = q\). \(\square\)

By (3.4), for \(R \in \mathcal{R}(A)\), we have
\[
\mathcal{B}^R_{x,y} = \{(y, x) \in A \times A : s^R_{x,y} \in \{q^+ : q \in Q_{>0}\}\}.
\]

4. The function \(\Phi_R\)

For \(R \in \mathcal{R}(A)\), denote \(\Phi^R : A \times A \to \mathbb{B}\) the function \((x, y) \mapsto t^R_{x,y}\). Following Section 3, for \(R \in \mathcal{R}(A)\) and \((x, y) \in A \times A\), we have \((x, y) \in A^R \iff \Phi^R(x, y) \in A\); in particular, we have
\[
R \in \mathcal{R}'(A) \iff \Phi^R(A \times A) \subseteq A.
\]  

Proposition 4.2. For \(R \in \mathcal{R}(A)\), \(\Phi^R\) is the unique function \(\Phi : A \times A \to \mathbb{B}\) satisfying (for all \(x, y \in A\) and all \(m, n \in \mathbb{N}^+\):

1. \(\Phi(mx, ny) = \Phi^n(R)\Phi(x, y);\)
2. \(xRy \iff \Phi(x, y) > 1.\)

Conversely, any binary relation \(R^R\) on \(A\) such that there exists a function \(\Phi^R : A \times A \to \mathbb{B}\) satisfying (1) and (2), belongs to \(\mathcal{R}(A)\).

Proof. The converse is straightforward, and for \(R \in \mathcal{R}(A)\), the function \(\Phi^R\) satisfies the conditions (1) and (2) of the proposition. Let \(R \in \mathcal{R}(A)\), and let \(\Phi, \Phi' : A \times A \to \mathbb{B}\) be two functions satisfying the conditions (1) and (2) of the proposition. Suppose that there exists a couple \((x, y) \in A \times A\) such that \(\Phi'(x, y) \neq \Phi(x, y)\). By symmetry, we can suppose that \(\Phi'(x, y) > \Phi(x, y)\). By Remark 1.1, there exist \(p, q \in \mathbb{N}^+\) such that \(pq^{-1}\Phi'(x, y) > 1 \geq pq^{-1}\Phi(x, y)\). Therefore \(pxRqy\) and \(qx(R)qy\); contradiction. Hence \(\Phi\) is unique. \(\square\)

The functions \(\Phi_{R^0}\) and \(\Phi_{R^\infty}\) are constant, given by
\[
\Phi_{R^0} = 0^+,
\]
\[
\Phi_{R^\infty} = \infty.
\]
And for \(R \in \mathcal{R}(A)\) and \(q \in Q_{>0}\), we have
\[
\Phi^R = q \cdot \Phi_R.
\]
For \(R \in \mathcal{R}(A)\), the function \(\Phi^R : A \times A \to A\) is given by
\[
\Phi^R(x, y) = \begin{cases} 
\Phi_R(x, y) & \text{if } \Phi_R(x, y) \in A \\
q & \text{if } \Phi_R(x, y) = q^+.
\end{cases}
\]
We thus have \(\Phi_R \geq \Phi^R\) with the equality if and only if \(R = R^\prime\). For \(R \in \mathcal{R}'(A)\), let
\[
\mathcal{R}(A)_R = \{S \in \mathcal{R}(A) : S^\prime = R\}
\]
be the fibre of the projection \(\mathcal{R}(A) \to \mathcal{R}'(A)\) above \(R\). We thus have
\[
\mathcal{R}(A)_R = \{S \in \mathcal{R}(A) : \{\Phi_S(x, y) \mid \Phi_S(x, y) \in A \times A\}, \forall (x, y) \in A \times A\}.
\]
Let \(R \in \mathcal{R}(A)\). Following (3.3), for \(x, y \in A\), we have \(xR^y y\) if and only if \(s^R_{x,y} > 1\). Define \(\sigma^R_{x,y} \in \mathbb{B}\) by
\[
\sigma^R_{x,y} = \begin{cases} 
\begin{cases} 
s^R_{y,x} & \text{if } s^R_{x,y} \in Q^+_{>0} \\
0 & \text{if } s^R_{x,y} = r^+\text{ with } r \in \mathbb{R}_{>0} \setminus Q_{>0}.
\end{cases}
\end{cases}
\]
Then we have $\sigma_{x,x}^R = \sigma_{x,x}^L = \frac{n}{m} \cdot \sigma_{x,x}^L$ and $xR^y y \iff \sigma_{y,x}^R > 1$. Therefore, the function $A \times A \to B$, $(x, y) \mapsto \sigma_{y,x}^R$ coincides with $\Phi_R$. And by (3.4), for $x, y \in A$, the triplet $(x, R^y, y)$ is super-Archimedean if and only $\sigma_{y,x}^R \not\in \{q^+ : q \in Q_{>0}\}; i.e. if and only if $t_{y,x}^R \not\in Q_{>0}$. We deduce that

$$R \in R'(A)^c \iff \Phi_R(A \times A) \subset B \setminus Q_{>0}.$$  \hspace{1cm} (4.3)

By (4.1) and (4.3), for $R \in R(A)$, we have

$$R \in R'(A) \iff \Phi_R(A \times A) \subset A \setminus Q_{>0}.$$  \hspace{1cm} (4.4)

Let $R^{\prec \succ}(A)$ be the set of relations $R \in R(A)$ such that $\Phi_R(A \times A) \subset [0^+, \infty)$. By (4.4), we have the inclusion $R^{\prec \succ}(A) \subset R''(A)$.

Precisely, the involution $R(A) \to R(A), R \mapsto R'$ induces by restriction a bijective map $R^{\prec \succ}(A) \to R^{\prec \succ}(A)$.

5. The functions $\Phi_>, \Phi_\geq : R_{>0} \times R_{>0} \to B$

The relations $>$ and $\geq$ on the $R_{>0}$-set $R_{>0}$, are positive homothetic orders, and we have $\geq = (>)^\wedge$. Let

$$Q = \{(r, r') \in R_{>0} \times R_{>0} : r^{-1}r' \in Q_{>0}\}.$$  

The subsets $B^>$ and $B^\geq$ of $R_{>0} \times R_{>0}$ are given by:

$$B^> = \{(r^+, r') : (r, r') \in Q\},$$  

$$B^\geq = B^> \cup Q \cup \{(r^+, r'^+) : (r, r') \in Q\}.$$  

And the subsets $B^\geq_1$ and $B^\geq_2$ $(i = 1, 2)$ of $R_{>0}$ are given by:

$$B^\geq_1 = \{(r^+ : r \in R_{>0}\},$$  

$$B^\geq_2 = R_{>0},$$  

$$B^\geq_2 = B^\geq_1 \cup B^\geq_2.$$  

Following (4.2), for $R \in \{>, \geq\}$, there exists a unique function $\Phi_R : R_{>0} \times R_{>0} \to B$ satisfying (for all $r, r' \in R_{>0}$ and all $m, n \in N^+$):

1. $\Phi_R(m \cdot r, n \cdot r') = \frac{m}{n} \cdot \Phi_R(r, r')$;
2. $r > r' \iff \Phi_R(r, r') > 1$.

The function $\Phi_>$ is explicitly given by:

$$\Phi_>(r, \infty) = \Phi_>(0^+, r) = 0^+ \quad (r \in R_{>0}^+),$$  

$$\Phi_>(\infty, r) = \infty \quad (r \in R_{>0} \setminus \{\infty\}),$$  

$$\Phi_>(r, 0^+) = \infty \quad (r \in R_{>0} \setminus \{0^+\}),$$  

$$\Phi_>(r, r') = \Phi_>(r^+, r'^+) = \Phi_>(r^+, r'^+) = r^{-1} - r \quad (r, r' \in R_{>0}),$$  

$$\Phi_>(r^+, r') = r - r^{-1} \quad (r, r' \in R_{>0}, r^{-1}r \not\in Q_{>0}),$$  

$$\Phi_>(r^+, r') = (r^{-1})^+ \quad (r, r' \in R_{>0}, r^{-1}r \in Q_{>0}).$$  

And the function $\Phi_\geq$ is explicitly given by:

$$\Phi_\geq(\infty, r) = \Phi_\geq(r, 0^+) = \infty \quad (r \in R_{>0}^+),$$  

$$\Phi_\geq(\infty, r) = 0^+ \quad (r \in R_{>0} \setminus \{\infty\}),$$  

$$\Phi_\geq(r, 0^+) = 0^+ \quad (r \in R_{>0} \setminus \{0^+\}),$$  

$$\Phi_\geq(r, r') = \Phi_\geq(r^+, r'^+) = \Phi_\geq(r^+, r'^+) = r^{-1} - r \quad (r, r' \in R_{>0}, r^{-1}r \not\in Q_{>0}),$$  

$$\Phi_\geq(r^+, r') = r - r^{-1} \quad (r, r' \in R_{>0}, r^{-1}r \not\in Q_{>0}),$$  

$$\Phi_\geq(r^+, r') = r^{-1} - r \quad (r, r' \in R_{>0}).$$  

By the above formulas, for $R \in \{>, \geq\}, r, r' \in R_{>0}$ and $a, b \in Q_{>0}$, we have

$$\Phi_R(a \cdot r, b \cdot r') = b^{-1}a \cdot \Phi_R(r, r').$$  \hspace{1cm} (5.1)

Remark 5.2. Thanks to the above formulas, for $R \in \{>, \geq\}$, we can explicitly describe the relation $R' \in R'(A)$.  

Remark 5.3. The relations $>$ and $\geq$ induce by restriction two positive homothetic orders on the $Q_{>0}$-set $B$. And the functions $B \times B \to B$ associated to these two orders are of course the restrictions to $B \times B$ of the functions $\Phi_>$ and $\Phi_\geq$. 

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6. Generalized homothetic biorders

Let $R$ be a binary relation on $A$. We say that $R$ is a generalized homothetic biorder if there exist two functions $u_1, u_2 : A \to \mathbb{R}_{\geq 0}^+$ satisfying (for all $x, y \in A$ and all $m \in \mathbb{N}^*$):

1. $u_i(mx) = m \cdot u_i(x)$ ($i = 1, 2$);
2. $x R y \iff u_1(x) > u_2(y)$.

Clearly, any generalized homothetic biorder on $A$ is an element of $\mathcal{R}(A)$. And the relations $R_0$ and $R_\infty$ are generalized homothetic biorders: for $R = R_0$, we can take for $(u_1, u_2)$ the pair of constant functions $(0^+, \infty)$; and for $R = R_\infty$, we can take for $(u_1, u_2)$ the pair of constant functions $(\infty, 0^+)$.

Let $\mathcal{R}_*(A)$ be the set of generalized homothetic biorders on $A$. And let

$$\mathcal{R}'_*(A) = \mathcal{R}_*(A) \cap \mathcal{R}'(A),$$
$$\mathcal{R}^{0, \infty}_*(A) = \mathcal{R}_*(A) \cap \mathcal{R}^{0, \infty}(A).$$

We thus have the inclusions

$$\{R_0, R_\infty\} \subset \mathcal{R}^{0, \infty}_*(A) \subset \mathcal{R}_*(A) \cap \mathcal{R}'(A) \subset \mathcal{R}'_*(A) \subset \mathcal{R}_*(A).$$

**Definition 6.1.** Let $R \in \mathcal{R}_*(A)$. We call representation of $R$ a pair of functions $(u_1, u_2)$ on $A$ with values in $\mathbb{R}_{\geq 0}^+$, satisfying the conditions (1) and (2) above. More generally, if $\mathcal{E}$ is a sub-$\mathbb{N}^*$-set of $\mathbb{R}_{\geq 0}^+$, we call representation of $R$ in $\mathcal{E}$ a pair of functions $(u_1, u_2)$ on $A$ with values in $\mathcal{E}$, satisfying the conditions (1) and (2) above.

**Lemma 6.2.** Let $R \in \mathcal{R}_*(A)$, and let $(u_1, u_2)$ be a representation of $R$. For all $x, y \in A$, we have

$$\Phi(x, y) = \Phi_*(u_1(x), u_2(y)).$$

**Proof.** Clear. □

For $R \in \mathcal{R}(A)$, we define

$$A_1^R = \{x \in A : x R y, \exists y \in A\} = \{x \in A : \mathcal{P}^R_{x,y} \neq \emptyset, \exists y \in A\},$$
$$A_2^R = \{y \in A : x R y, \exists x \in A\} = \{y \in A : \mathcal{P}^R_{x,y} \neq \emptyset, \exists x \in A\},$$
and

$$A_{1,2}^R = \{x \in A_1^R : y R^x y, \exists y \in A\} = \{x \in A_1^R : \mathcal{P}^R_{x,y} \neq \emptyset, \exists y \in A\},$$
$$A_{2,1}^R = \{y \in A_2^R : y R^x x, \exists x \in A\} = \{y \in A_2^R : \mathcal{P}^R_{x,y} \neq \emptyset, \exists x \in A\}.$$

We have the inclusion

$$\mathcal{B}^R \subset A_{1,2}^R \times A_{2,1}^R.$$

Notice that

$$A_1^R = \emptyset \iff A_2^R = \emptyset \iff R = R_0$$
and that

$$A_{1,2}^R = \emptyset \iff A_{2,1}^R = \emptyset \iff R \in \mathcal{R}^{0, \infty}(A).$$

Let $R \in \mathcal{R}_*(A)$, and let $(u_1, u_2)$ be a representation of $R$. Then we have

$$\mathcal{B}^R = \{(x, y) \in A \times A : (u_1(x), u_2(y)) \in \mathcal{B}^\ast\}.$$

We then deduce that

$$\mathcal{B}_1^R = \{x \in A : u_1(x) = u_2(y)^+, \exists y \in A\},$$
$$\mathcal{B}_2^R = \{y \in A : u_1(x) = u_2(y)^+, \exists x \in A\}.$$

If $R \neq R_0$, then $u_1 \neq 0^+$ and $u_2 \neq \infty$, and we have

$$A_1^R = \{x \in A : u_1(x) \neq 0^+\},$$
$$A_2^R = \{y \in A : u_2(y) \neq \infty\}.$$

And if $R \notin \mathcal{R}^{0, \infty}(A)$, then $u_i(A) \notin [0^+, \infty)$ ($i = 1, 2$), and we have

$$A_{1,2}^R = \{x \in A : u_1(x) \notin [0^+, \infty]\}.$$
Lemma 6.4. Let $R \in \mathcal{R}_{\infty}^{\infty}(A) \setminus \{R_0\}$. There exists a representation $(u_1, u_2)$ of $R$ in $[0^+, \infty)$, and this representation is unique.

Proof. For $(x, y) \in A \times A$, we define

$$u_1(x) = \begin{cases} \infty & \text{if } x \in A_1^R \\ 0^+ & \text{otherwise} \end{cases}$$

and

$$u_2(y) = \begin{cases} 0^+ & \text{if } y \in A_2^R \\ \infty & \text{otherwise}. \end{cases}$$

Since $R \in \mathcal{R}_{\infty}^{\infty}(A)$, we have $xRy \iff (x, y) \in A_1^R \times A_2^R$. Therefore, the pair $(u_1, u_2)$ represents $R$. And since $R \neq R_0$, the sets $A_1^R$ and $A_2^R$ are nonempty, therefore $(u_1, u_2)$ is the unique representation of $R$ in $[0^+, \infty)$. □

Notice that in (6.4), without the condition $u_i(A) \subseteq [0^+, \infty)$ ($i = 1, 2$), the uniqueness property is no longer true: for any two morphisms of $\mathbb{N}^+$-sets $u_1 : A \to A \setminus [0^+]$ and $u_2 : A \to A \setminus [\infty]$, the pairs $(u_1, 0^+) \quad \text{and} \quad (\infty, u_2)$ represent $R_0$. Note also that for $R = R_0$, the Lemma 6.4 is not true: the pairs $(0^+, 0^+)$, $(0^+, \infty)$ and $(\infty, \infty)$ represent $R_0$.

Lemma 6.5. Let $R \in \mathcal{R}_{\infty}^{\infty}(A) \setminus \mathcal{R}_{\infty}^{\infty}(A)$. There exists a representation $(u_1, u_2)$ of $R$ such that for $i = 1, 2$, we have $u_i(A_1^R) \subseteq A_i$.

In particular, we have $u_2(A) \subseteq A$, and if $R \in \mathcal{R}(A)$, then $(u_1, u_2)$ is a representation of $R$ in $A$. Moreover, up to multiplication by an element of $\mathbb{R}_{>0}$, the pair $(u_1, u_2)$ is unique: if $(u_1', u_2')$ is another representation of $R$ such that for $i = 1, 2$, we have $u_i'(A_1^R) \subseteq A_i$, then there exists a $\lambda \in \mathbb{R}_{>0}$ such that $(u_1', u_2') = (\lambda \cdot u_1, \lambda \cdot u_2)$.

Proof. Let $(u_1, u_2)$ be a representation of $R$. For $i = 1, 2$, let $u_i : A \to A$ be the function defined by

$$u_i(x) = \begin{cases} u_i(x) & \text{if } u_i(x) \in A_i \text{ or } x \in A_2^R \\ 0^+ & \text{if } u_i(x) = r^+ \text{ and } x \in A_1^R. \end{cases}$$

Since $(x, y) \in A \times A : u_1(x) = u_2(y)^+ \subseteq B_1^R \times B_2^R$, the pair $(u_1, u_2)$ is a representation of $R$. By construction, for $i = 1, 2$, we have $u_i(A_1^R) \subseteq A_i$. And since the set $\{y \in A : u_2(y) \in B \setminus A\}$ is contained in $A_2^R$, we have $u_2(A) \subseteq A$. Finally, if $R \in \mathcal{R}(A)$, since $A_1^R = A = A_2^R$, $(u_1, u_2)$ is a representation of $R$ in $A$.

Let $(u_1', u_2')$ be another representation of $R$ such that for $i = 1, 2$, we have $u_i'(A_1^R) \subseteq A_i$. By (6.2), for $x, y \in A$, we have

$$\Phi_2(u_1(x), u_2(y)) = \Phi_2(x, y) = \Phi_2(u_1'(x), u_2'(y)).$$

Since $R \not\subseteq \mathcal{R}_{\infty}^{\infty}(A)$, we have $u_i(A) \not\subseteq [0^+, \infty)$ ($i = 1, 2$), and hence:

- on $A \setminus A_1^R$, we have $u_1 = u_1' = 0^+$;
- on $A_1^R \setminus A_1^R$, we have $u_1 = u_1' = \infty$;
- on $A \setminus A_2^R$, we have $u_2 = u_2' = \infty$;
- on $A_2^R \setminus A_2^R$, we have $u_2 = u_2' = 0^+$.

On the other hand, for $(x, y) \in (A_1^R \times A_2^R) \cap (A_1^R \times A_2^R)$, we have $u_1(x), u_2(y), u_1'(x), u_2'(y) \in \mathbb{R}_{>0}$ and

$$u_2'(y)^- u_1(x) = \Phi_2(x, y) = u_2(y)^- u_1(x).$$

Therefore, if $A_1^R \times A_2^R = \emptyset$, then there exists a constant $\lambda \in \mathbb{R}_{>0}$ such that for all $(x, y) \in A_1^R \times A_2^R$, we have

$$(u_1'(x), u_2'(y)) = (\lambda \cdot u_1(x), \lambda \cdot u_2(y)).$$

In particular, if $R \in \mathcal{R}(A)$, the lemma is proved.

Suppose now that $R \not\subseteq \mathcal{R}(A)$. Then the set $B^R = \Phi_2^{-1}(B \setminus A)$ is nonempty. And for $(x, y) \in B^R$, we have $u_1(x) \in B_{>0} \setminus A$, $u_2(y) \in \mathbb{R}_{>0}$, and $u_2(y)^- u_1(x) = q^+$ for a $q \in \mathbb{Q}_{>0}$; in particular, we have $\Phi_2(x, y) = u_2(y)^- u_1(x)$. For $(x, y) \in B^R$, we thus have

$$u_2(y)^- u_1(x) = u_2'(y)^- u_1'(x) \in \{q^+ : q \in \mathbb{Q}_{>0}\}.$$ 

Therefore, there exists a constant $\mu \in \mathbb{R}_{>0}$ such that for all $(x, y) \in B^R$, we have

$$(u_1(x), u_2(y)) = (\mu \cdot u_1(x), \mu \cdot u_2(y)).$$

In particular, if $\Phi_2^{-1}(\mathbb{R}_{>0}) = \emptyset$, the lemma is proved.

Take a couple $(x, y) \in \Phi_2^{-1}(\mathbb{R}_{>0}) \subseteq A_1^R$, and let us show that $u_1'(x) = \mu \cdot u_1(x)$ and $u_2'(y) = \mu \cdot u_2(y)$. If $x \in B_1^R$, then there exists a $b \in A$ such that $(x, b) \in B_1^R$; and we have $u_1'(x) = \mu \cdot u_1(x)$. Suppose that $x \in A_1^R$. Then $u_1(x), u_1'(x) \in \mathbb{R}_{>0}$. If $u_1(x) > \mu u_1(x)$, then there exist $m, n \in \mathbb{N}^+$ such that

$$u_1(mx) > u_2(nb) = \mu u_2(nb) > \mu u_1(mx).$$

contradiction. Also, if \( u'_1(x) < \mu u_1(x) \), then there exist \( m, n \in \mathbb{N}^+ \) such that
\[
u'_1(mx) < u'_2(nb) = \mu u_2(nb) < \mu u_1(mx);
\]
contradiction. Hence \( u'_1(x) = \mu u_1(x) \). The equality \( u'_2(y) = \mu \cdot u_2(y) \) is obtained similarly. This ends the proof of the lemma. \( \square \)

Notice that in (6.5), it follows from the above proof that without the condition \( u_i(A^R) \subset A \) \((i = 1, 2)\), the uniqueness property is no longer true.

**Lemma 6.6.** Let \( R \in \mathcal{R}_*(A) \), and let \( (u_1, u_2) \) be a representation of \( R \). We have:
\[
R \in \mathcal{R}^{R,\infty}_*(A) \iff u_1(A) \subset [0^+, \infty) \text{ or } u_2(A) \subset [0^+, \infty),
\]
\[
R \in \mathcal{R}^+_*(A) \iff u_1(A) \cap \{r^+ : r \in u_2(A) \cap \mathbb{R}_0^+\} = \emptyset.
\]

**Proof.** For \((x, y) \in A \times A\), we have \( \Phi_R(x, y) \in [0^+, \infty) \) if and only if \( u_1(x) \in [0^+, \infty) \) or \( u_2(y) \in [0^+, \infty) \); and we have \((x, y) \in \mathcal{B}^R \) if and only there exists a \( q \in \mathbb{Q}_{>0} \) such that \( u_1(x) = q \cdot u_2(y)^+ \). This ends the proof of the lemma. \( \square \)

7. **Generalized homothetic intervals (resp. weak) orders**

A generalized homothetic biorder on \( A \) is called a:
- a **generalized homothetic interval order** if for any (i.e. for one) representation \((u_1, u_2)\) of \( R \), we have \( u_1 \leq u_2 \);
- a **generalized homothetic weak order** if for any (i.e. for one) representation \((u_1, u_2)\) of \( R \), we have \( u_1 = u_2 \); in which case we say that \( u_1 \) is a representation of \( R \).

Let us recall that a relation binary \( R \) on \( A \) is said to be:
- reflexive if for all \( x \in A \), we have \( xRx \);
- symmetric if for all \( x, y \in A \), we have \( xRy \iff yRx \);
- transitive if for all \( x, y, z \in A \), we have \( xRyRz \Rightarrow xRz \).

For all binary relations \( R \) and \( R' \) on \( A \), we note \( R \cap R' \) the binary relation on \( A \) defined by
\[
x(R \cap R')y \iff xRy \text{ and } yR'x.
\]

Note that for any binary relation \( R \) on \( A \), the **indifference relation** \( S = R^\cap \setminus (R) \) is symmetric. Indeed, for \( x, y \in A \), we have
\[
xSy \iff x(-R)y \text{ and } y(-R)x.
\]

**Lemma 7.1.** Let \( R \in \mathcal{R}_*(A) \), and let \( S = R^\cap \setminus (R) \). Then \( R \) is:
- a **generalized homothetic interval order** if and only if \( S \) is reflexive;
- a **generalized homothetic weak order** if and only if \( S \) is reflexive and transitive.

**Proof.** Let \((u_1, u_2)\) be a representation of \( R \). For \( x, y \in A \), we have
\[
xSy \iff \begin{cases} u_2(x) \geq u_1(y) \\ u_2(y) \geq u_1(x) \end{cases}.
\]

Therefore, \( S \) is reflexive if and only if \( u_1 \leq u_2 \); i.e. if and only if \( R \) is a generalized homothetic interval order.

Suppose that \( R \) is a generalized homothetic weak order. Then \( u_1 = u_2 \), and for all \( x, y \in A \), we have \( xSy \iff u_1(x) = u_1(y) \).

Therefore \( S \) is transitive.

Conversely, suppose that \( S \) is reflexive and transitive. Suppose that there exists \( x \in A \) such that \( u_1(x) \neq u_2(x) \). Since \( xSy \), we have \( u_1(x) < u_2(x) \). Let \( q, q' \in \mathbb{Q}_{>0} \) such that \( q < 1 < q' \) and
\[
qu_1(x) < u_1(x) < qu_2(x) < qu_2(x) < u_2(x) < qu_2(x).
\]

Write \( q = \frac{m}{n} \) and \( q' = \frac{m'}{n'} \) with \( m, n, m', n' \in \mathbb{N}^+ \), and let \( y = mn'x, z = mn'x \) and \( t = m'nx \). Then we have
\[
u_1(z) < u_1(y) < u_2(z) < u_1(t) < u_2(y) < u_2(t).
\]

Hence \( zSyS \) and \( z(-S) \); contradiction. Therefore, \( u_1 = u_2 \). \( \square \)

**Remark 7.2.** The relation \( > \) on \( \mathbb{R}_0^+ \) is a generalized homothetic weak order: it is represented by the identity morphism \( \mathbb{R}_0^+ \to \mathbb{R}_0^+ \).

**Remark 7.3.** The empty relation on \( A \) is a generalized homothetic weak order: the constant functions \( u = 0^+ \) and \( u = \infty \) represent \( R_0 \). On the contrary, the trivial relation on \( A \) is **neither** a generalized homothetic weak order nor a generalized homothetic interval order: for all representations \((u_1, u_2)\) of \( R_\infty \), we have \( u_1 > u_2 \). \( \star \)
8. An example: The relation $\succ$ on $T(A)$

Let $T(A) = A \times R(A) \times A$. We endow $T(A)$ with the structure of a $\mathbb{Q}_{\geq 0}$-set (hence a fortiori of a $\mathbb{N}^*$-set) defined by $q \cdot (x, y, z) = (x, R^q y, z)$, and we note $\succ$ the binary relation on $T(A)$ defined by: $(x_1, R_1, y_1) \succ (x_2, R_2, y_2)$ if and only there exist $m, n \in \mathbb{N}^*$ such that $mx_1 R_1 ny_1$ and $mx_2 R_2 ny_2$.

**Lemma 8.1.** The relation $\succ$ on $T(A)$ is a generalized homothetic weak order. Moreover, the function $u_\succ : T(A) \to \mathbb{B}$ given by $u_\succ (x, y, z) = \Phi_R(x, y)$ is a representation of $\succ$ in $\mathbb{B}$, and a morphism of $\mathbb{Q}_{\geq 0}$-sets.

**Proof.** The function $\Phi$ is clearly a morphism of $\mathbb{Q}_{\geq 0}$-sets. Let $(x_1, R_1, y_1), (x_2, R_2, y_2) \in T(A)$. We have $(x_1, R_1, y_1) \succ (x_2, R_2, y_2)$ if and only if there exist $m, n \in \mathbb{N}^*$ such that $\frac{m}{n} \cdot \Phi_{R_1}(x, y) > 1 \geq \frac{m}{n} \cdot \Phi_{R_2}(x, y)$; i.e., if and only there exists a $q \in \mathbb{Q}_{\geq 0}$ such that $\Phi_{R_1}(x, y) > q \geq \Phi_{R_2}(x, y)$. By (1.1), we obtain that

$$(x_1, R_1, y_1) \succ (x_2, R_2, y_2) \iff \Phi_{R_1}(x, y) > \Phi_{R_2}(x, y).$$

Hence the lemma. □

Let $\succeq$ the binary relation on $T(A)$ defined by $\succeq = \succ \vee$. It is given by

$$(x_1, R_1, y_1) \succeq (x_2, R_2, y_2) \iff u_\succ (x_1, R_1, y_1) \geq u_\succ (x_2, R_2, y_2).$$

Let $\sim$ be the indifference relation associated with $\succ$, defined by

$$(x_1, R_1, y_1) \sim (x_2, R_2, y_2) \iff u_\succ (x_1, R_1, y_1) = u_\succ (x_2, R_2, y_2).$$

This is an equivalence relation on $T(A)$. The quotient set $\overline{T}(A) = T(A)/\sim$ inherits the $\mathbb{Q}_{\geq 0}$-set structure of $T(A)$, and $u_\succ : T(A) \to \mathbb{B}$ is factorized through an injective morphism of $\mathbb{Q}_{\geq 0}$-sets

$$\overline{u}_\succ : \overline{T}(A) \to \mathbb{B}.$$ 

The study of the properties of this morphism will be the subject of a further work.

9. Characterization of generalized homothetic biorders

Consider the six following properties (for all $x, y, z, t \in A$):

\((1)\) if $(x, t) \in A_R$ and $x R y R^t z \in R$, then we have $x R t$;

\((2)\) if $(x, t) \in A_R$, $(y, z) \in A_{R_1} \times A_{R_2}$ and $x R t$, then there exist $m, n, p \in \mathbb{N}^*$ such that we have $mx R ny R^t p z R mt$;

\((3)\) if $(x, t) \in B_R$ and $t R^y z R^x R$, then we have $t R^x x$;

\((4)\) if $(x, t) \in B_R$, $(y, z) \in A_{R_1} \times A_{R_2}$ and $t R^y x$, then there exist $m, n, p \in \mathbb{N}^*$ such that we have $mt R^y ny R p z R^t mx$;

\((5)\) The fibre $\Phi_R^{-1}(0^+)$ is empty or union of sets of the form $\{x\} \times A$ or $A \times \{y\}$, and the fibre $\Phi_R^{-1}(\infty)$ is empty or union of sets of the form $\{x\} \times A_{R_2}$ or $A_{R_1} \times \{y\}$;

\((6)\) if $(x, y) \in B_{R_1} \times B_{R_2}$, then we have $(y, x) \in A_{R} R'^{x}$.

**Remark 9.1.** In the condition \((6)\), we can replace the set $B_{R_1} \times B_{R_2}$ by the set $(B_{R_1} \times B_{R_2}) \cap A_R$. Indeed, by (3.4), we know that for $(x, y) \in B_R$, we have $(y, x) \in A_{R'} R^x$. For $(x, y) \in B_R \times B_{R'}$, the triplet $(x, R, y)$ is potentially non super-Archimedean in the sense that there exist $x', y' \in A$ such that the triplets $(x', R, y')$ and $(x', R, y)$ are non super-Archimedean. And the condition \((6)\) means that the triplets potentially non super-Archimedean $(x, R, y)$ behave as “true” non super-Archimedean triplets. □

**Lemma 9.2.** Let $R \in \mathcal{R}(A)$. If $R$ satisfies \((5)\) and \((6)\), then we have

$$\Phi_R^{-1}(\mathbb{B} \setminus \{0^+, \infty\}) = A_{R_1} \times A_{R_2}.$$

**Proof.** For $x, y \in A$, we have

\begin{align*}
  x & \in A \times A_1 \iff \Phi_R(x, A) = 0^+; \\
  y & \in A \times A_2 \iff \Phi_R(A, y) = 0^+,
\end{align*}

and

\begin{align*}
  x & \in A_1 \times A_{R_1} \iff \Phi_R(x, A) = \infty; \\
  y & \in A_2 \times A_{R_2} \iff \Phi_R(A, y) = \infty.
\end{align*}
If $R$ satisfies ($S$), we deduce that

\[ \Phi_R(x, y) = 0^+ \iff x \in A \setminus A_R^r \text{ or } y \in A \setminus A_R^s, \]
\[ \Phi_R(x, y) = \infty \iff x \in A_R^r \setminus A_{1,2}^r \text{ or } y \in A_R^s \setminus A_{2,1}^s. \]

Hence the lemma. \[ \square \]

**Proposition 9.3.** For $R \in \mathcal{R}(A)$, we have $R \in \mathcal{R}_*(A)$ if and only if $R$ satisfies the properties ($i$) for $i = 1, \ldots, 6$.

**Proof.** Let $R \in \mathcal{R}_*(A)$, and let $(u_1, u_2)$ be a representation of $R$. Then the relation $R^\vee$ on $A$ is given by

\[ xR^\vee y \iff u_2(x) \geq u_1(y). \]

By looking at description of the sets $\mathcal{B}_R^s, A_R^1$ and $A_R^2$ given in the Section 6, is easy to verify that the properties ($1S$), ($2S$), ($3S$) and ($4S$), are true for $R$. For $x, y \in A$, we have $\Phi_R(x, y) = 0^+$ if and only if $u_1(x) = 0$ or $u_2(x) = \infty$; and we have $\Phi_R(x, y) = \infty$ if and only if one of the two following conditions is satisfied:
- $u_2(x) = \infty$ and $u_1(y) \neq \infty$;
- $u_1(x) \neq 0^+$ and $u_2(y) = 0^+$.

Therefore $R$ satisfies ($S$). As for the property ($6S$), let $(x, y) \in \mathcal{B}_R^s \times \mathcal{B}_R^s$. By Remark 9.1, we can suppose that $(x, y) \in \mathcal{A}_R$. By Section 6, we have $u_1(x) = r^+$ and $u_2(y) = r'$ for some $r, r' \in \mathbb{R}_>0$. By (3.4), we have $(x, y) \in \mathcal{A}_R^s$ if and only if $t_{x,y}^s \in \mathbb{R}_>0 \setminus \mathbb{Q}_>0$. But by (6.2) and the Section 5, we have

\[ t_{x,y}^s = r^{-1}r \begin{cases} r^{-1}r & \text{if } r^{-1}r \in \mathbb{R}_>0 \setminus \mathbb{Q}_>0 \vphantom{\infty_2} \\ (r^{-1}r)^+ & \text{if } r^{-1}r \in \mathbb{Q}_>0. \end{cases} \]

But the case $r^{-1}r \in \mathbb{Q}_>0$ is not possible, because $(x, y) \in \mathcal{A}_R$. Hence $R$ satisfies ($S$).

Conversely, let $R \in \mathcal{R}(A)$ be a relation satisfying the properties ($i$) for $i = 1, \ldots, 6$. We can suppose that $R \neq R_0$. Then $A_R^1 \neq \emptyset$ and $A_R^2 \neq \emptyset$. By ($S$), for $(x, y) \in \Phi_R^{-1}([0^+, \infty))$, we can let

\[ u_1(x) = \begin{cases} 0^+ & \text{if } \Phi_R(x, A) = 0^+ \\ \infty & \text{if } \Phi_R(x, A) = \infty \end{cases} \]

and

\[ u_2(y) = \begin{cases} 0^+ & \text{if } \Phi_R(A, y) = 0^+ \\ \infty & \text{if } \Phi_R(A, y) = \infty. \end{cases} \]

The function $u_1 \times u_2$ on $\Phi_R^{-1}([0^+, \infty))$ is well-defined, and for $(x, y) \in \Phi_R^{-1}([0^+, \infty))$, we have

\[ xR^\vee y \iff u_1(x) = \infty > 0^+ = u_2(y). \]

In particular if $R \in \mathcal{R}_t(A)$, then $\Phi_R^{-1}([0^+, \infty)) = A \times A$, the functions $u_1, u_2 : A \to [0^+, \infty]$ are morphisms of $\mathbb{N}_*$-sets, and the relation $R$ is a generalized homothetic biorder.

We now suppose that $R \notin \mathcal{R}_t(A)$. Then $A_R^1, A_R^2 \neq \emptyset$ and $t_{x,y}^s \neq \emptyset$. By (9.2), we have

\[ \Phi_R^{-1}([0^+, \infty)) = A_{1,2}^r \times A_{2,1}^r. \]

We hence need to extend the function $u_1 \times u_2$ on $A_{1,2}^r \times A_{2,1}^r$. Let a couple $(a, b) \in A_{1,2}^r \times A_{2,1}^r$. For $(x, y) \in A_R$, let $A_R^s$ (1S), and $A_R^s$ (2S), we have the equality (cf. [13] Lemma 3.4)

\[ \mathcal{G}_{x,y}^R = \mathcal{G}_{x,0}^R \mathcal{G}_{0,y}^R \mathcal{G}_{0,0}^R. \]

Also, for $(x, y) \in \mathcal{B}_R$, (3S) and (4S), as in the proof of Lemma 3.4 of [13], we obtain the equality

\[ \mathcal{G}_{x,y}^R = \mathcal{G}_{x,0}^R \mathcal{G}_{0,y}^R \mathcal{G}_{0,0}^R \quad (**). \]

Suppose first that $R \in \mathcal{R}_t(A) \setminus \mathcal{R}_t(A)$. Then $(a, b) \in A_R^s$, therefore $s_{a,b}^R = r^+$ for $r \in \mathbb{R}_>0$; and by (3.4), we have

\[ t_{x,a}^s = \begin{cases} r & \text{if } r \in \mathbb{R}_>0 \setminus \mathbb{Q}_>0 \\ r^+ & \text{if } r \in \mathbb{Q}_>0. \end{cases} \]

In particular, $(b, a)$ is an element of $A_{1,2}^s \times A_{2,1}^s$. Let $(x, y) \in A_{1,2}^s \times A_{2,1}^s$. Since $(x, b)$ and $(a, x)$ are elements of $A_{1,2}^s \times A_{2,1}^s$, $t_{x,b}^R$ and $t_{x,y}^R$ are elements of $\mathbb{R}_>0$, and by (**), we have the equality

\[ t_{x,y}^R = t_{x,b}^R t_{b,y}^R. \]
Let
\[ u_1(x) = t_{x,b}^R \]
and
\[ u_2(y) = r^{-1}(t_{a,y}^R)^{-1}. \]
Then we have
\[ x R y \Leftrightarrow t_{x,y}^R > 1 \Leftrightarrow u_1(x) > u_2(y). \]
The functions \( u_1, u_2 : A \rightarrow A \) thereby defined are morphisms of \( \mathbb{N}^n \)-modules, and the relation \( R \) is a generalized homothetic boider.

Suppose now that \( R \in \mathcal{R}(A) \smallsetminus \mathcal{R}'(A) \). Then \( \mathcal{B}^R (\subset A_{1,2} \times A_{2,1}^R) \neq \emptyset \), and we can suppose that the pair \((a, b)\) has been chosen such that:
- \((a, b) \in \mathcal{A}^R \) if the inclusion \( \mathcal{B}^R \subset A_{1,2} \times A_{2,1}^R \) is strict;
- \( t_{a,b}^R \in \mathbb{Q}_{>0} \) if the set \( \{(a', b') \in \mathcal{A}^R : t_{a,b'}^R \in \mathbb{Q}_{>0}\} \) is nonempty.

By (3.4), three cases may appear:
- **case 1**: \( t_{b,a}^R = s_{a,b}^R = q \in \mathbb{Q}_{>0} \) if \((a, b) \in \mathcal{B}^R \);
- **case 2**: \( t_{b,a}^R = (t_{b,a}^R)^{-1} = r \in \mathbb{R}_{>0} \smallsetminus \mathbb{Q}_{>0} \) if \((a, b) \in \mathcal{A}^R \) and \( t_{a,b}^R \notin \mathbb{Q}_{>0} \);
- **case 3**: \( t_{b,a}^R = s_{a,b}^R = q^+ \in \mathbb{B} \smallsetminus \mathcal{A} \) if \((a, b) \in \mathcal{A}^R \) and \( t_{a,b}^R \in \mathbb{Q}_{>0} \).

Denote \( \mathbb{R}_{>0}^\alpha \rightarrow \mathcal{A}, r \rightarrow \tilde{r} \) the projection defined by
\[ \tilde{r} = \begin{cases} r & \text{if } r \in \mathcal{A} \\ s & \text{if } r = s^+ \text{ for a } s \in \mathbb{R}_{>0}. \end{cases} \]

By (3.4), for \((x, y) \in A_{1,2} \times A_{2,1}^R \), we have \( t_{x,y}^{R'} = t_{x,y}^R \). Take \((x, y) \in A_{1,2} \times A_{2,1}^R \), let \( \alpha = t_{x,y}^R, \beta = s_{x,y}^R \). By (***), if \((x, y) \in \mathcal{A}^R \) (which excludes case 1), we have
\[ t_{x,y}^R = \begin{cases} r \tilde{\alpha} \tilde{\beta}^{-1} & \text{in case 2} \\ q \tilde{\alpha} \tilde{\beta}^{-1} & \text{in case 3}. \end{cases} \]

And by (****), if \((x, y) \in \mathcal{B}^R \), we have
\[ t_{x,y}^R = \begin{cases} q^+ \cdot \tilde{\alpha} \tilde{\beta}^{-1} & \text{in case 1} \\ r^+ \cdot \tilde{\alpha} \tilde{\beta}^{-1} & \text{in case 2} \\ q^+ \cdot \tilde{\alpha} \tilde{\beta}^{-1} & \text{in case 3}. \end{cases} \]

Let us show that
\[ t_{x,y}^R > 1 \Leftrightarrow \begin{cases} q^+ \cdot \tilde{\alpha} > \tilde{\beta} & \text{in case 1} \\ r^+ \cdot \tilde{\alpha} > \tilde{\beta} & \text{in case 2} \\ q \tilde{\alpha} > \tilde{\beta} & \text{in case 3 if } x \in \mathcal{A}^R_1 \text{ or } y \in \mathcal{A}^R_2 \\ q^+ \cdot \tilde{\alpha} > \tilde{\beta} & \text{in case 3 if } x \in \mathcal{A}^R_1 \times \mathcal{A}^R_2. \end{cases} \]

Case 1 is obvious.
In case 2, if \((x, y) \in \mathcal{A}^R \), we have \( t_{x,y}^R > 1 \Leftrightarrow r \tilde{\alpha} > \tilde{\beta} \); and if \((x, y) \in \mathcal{A}^R \) and \( r \tilde{\alpha} = \tilde{\beta} \), then \( t_{x,y}^R = 1 \), which is impossible (since we are in case 2).

Suppose that we are in case 3. If \((x, y) \in \mathcal{B}^R \subset \mathcal{B}^R_1 \times \mathcal{B}^R_2 \), we have \( t_{x,y}^R > 1 \Leftrightarrow q^+ \cdot \tilde{\alpha} > \tilde{\beta} \). Suppose then that \((x, y) \in \mathcal{A}^R \).

If \( x \in \mathcal{A}^R_1 \) or \( y \in \mathcal{A}^R_2 \), then we have \( t_{x,y}^R > 1 \Leftrightarrow q \tilde{\alpha} > \tilde{\beta} \). There remains the case \((x, y) \in \mathcal{B}^R_1 \times \mathcal{B}^R_2 \). This is when we use property \((6S)\): we have \( t_{x,y}^R > 1 \Leftrightarrow q \tilde{\alpha} > \tilde{\beta} \); and if \( q \tilde{\alpha} = \tilde{\beta} \), then \( t_{x,y}^R = q \tilde{\alpha} \tilde{\beta}^{-1} = 1 \in \mathbb{Q}_{>0} \), therefore by (3.4) \((x, y) \notin \mathcal{A}^R \), which contradicts property \((6S)\). We thus have \( t_{x,y}^R > 1 \Leftrightarrow q^+ \cdot \tilde{\alpha} > \tilde{\beta} \).

Let
\[ u_1(x) = \begin{cases} q^+ \cdot \tilde{t}_{x,b}^R & \text{in case 1} \\ r^+ \cdot \tilde{t}_{x,b}^R & \text{in case 2} \\ q \tilde{t}_{x,b}^R & \text{in case 3 if } x \in \mathcal{A}^R_1 \\ q^+ \cdot \tilde{t}_{x,b}^R & \text{in case 3 if } x \in \mathcal{B}^R_1 \end{cases} \]
and
\[ u_2(y) = \tilde{z}_{a,y}^R. \]
Then we have
\[ xRy \Leftrightarrow t_{x,y}^R \Leftrightarrow u_1(x) > u_2(y). \]
The functions \( u_1, u_2 : A \to A \) thereby defined are morphisms of \( \mathbb{N}^+ \)-modules, and the relation \( R \) is a generalized homothetic biorder. This ends the proof of the proposition.

\[ \Box \]

**Remark 9.4.** For \( R \in \mathcal{R}(A) \), the properties (3S), (4S) and (6S) are empty. Therefore, properties (1S), (2S) and (5S) characterize the relations \( R \in \mathcal{R}_*(A) \).

**Remark 9.5.** In general, the inclusion \( \mathcal{R}_*(A) \subset \mathcal{R}(A) \) is strict. For instance, take for \( A \) the union \( \mathbb{N}^+ \times \mathbb{N}^+ \) of two copies of \( \mathbb{N}^+ \), endowed with the natural structure of \( \mathbb{N}^+ \)-set, and let \( R \) stands for the binary relation on \( A \) defined by (for \( m, n \in \mathbb{N}^+ \)):

\[ - mRxnx \Leftrightarrow m > n; \]
\[ - mxny \Leftrightarrow m > n; \]
\[ - myny \Leftrightarrow m > n; \]
\[ - my(-R)nx \Leftrightarrow m > n. \]

The relation \( R \) is \( h \)-independent and \( h \)-positive, but it is not a generalized homothetic biorder.

**Remark 9.6.** The positive homothetic order \( \geq \) on \( \mathbb{R}^2_{\geq 0} \) is not a generalized homothetic biorder. Indeed, the property (6S) is not satisfied: for \( r, r' \in \mathbb{R}_{>0} \), we have \( (r', r^+) \in (\mathcal{B}_1^R \times \mathcal{B}_2^R) \setminus \mathcal{B}^R \) and \( (r^+, r') \in \mathcal{B} > \).

**10. “Operations” on generalized homothetic biorders**

Let us consider the projection \( \mathbb{R}^2_{\geq 0} \to \Lambda, r \to \tilde{r} \) defined in the proof of (9.3). And for any function \( u : A \to \mathbb{R}^2_{>0} \), denote \( \tilde{u} : A \to \Lambda \) the function defined by \( \tilde{u}(x) = u(x) \).

Let \( R \in \mathcal{R}_*(A) \), and let \((u_1, u_2)\) be a representation of \( R \). For \( q \in \mathbb{Q}_{>0} \), the positive homothetic order \( R^q \) is a generalized homothetic biorder represented by the pair of functions \((q \cdot u_1, u_2)\). Similarly, the positive homothetic order \( R^q \) is a generalized homothetic biorder represented by the pair of functions \((u_1, q \cdot u_2)\). As for the order \( R^r \), for \( x, y \in A \), we have
\[ xR^r y \Leftrightarrow u_2(x) \geq u_1(y). \]

**Lemma 10.1.** Let \( R \in \mathcal{R}_*(A) \), and let \((u_1, u_2)\) be a representation of \( R \). For all \( x, y \in A \), we have:
\[ \Phi_{R^r}(x, y) = \Phi_{R^r}(u_2(x), u_1(y)). \]

**Proof.** Clear. \[ \Box \]

Note \( \mathcal{R}_*(A)^\vee \) the subset of \( \mathcal{R}(A) \) formed by orders \( R \) such that there exist two functions \( v_1, v_2 : A \to \mathbb{R}^2_{>0} \) satisfying (for all \( x, y \in A \) and all \( m \in \mathbb{N}^+ \)):

1. \( v_i(mx) = mv_i(x) \) \( (i = 1, 2) \);
2. \( xRy \Leftrightarrow v_1(x) \geq v_2(y) \).

Let \( R \in \mathcal{R}_*(A)^\vee \), and let \((u_1, u_2)\) be a representation of \( R^\vee \). Then we have
\[ \mathcal{B}^R = \{(x, y) \in A \times A : (u_2(x), u_1(y)) \in \mathcal{B}^R \}. \]

The involution \( \mathcal{R}(A) \to \mathcal{R}(A), R \mapsto R^\vee \) induces by restriction two bijective maps
\[ \mathcal{R}_*(A) \to \mathcal{R}_*(A)^\vee, \]
\[ \mathcal{R}_*(A)^\vee \to \mathcal{R}_*(A), \]
that are inverse one another.

**Remark 10.2.** Directly or through the bijection \( \mathcal{R}_*(A) \to \mathcal{R}_*(A)^\vee \), one can characterize the relations \( R \in \mathcal{R}_*(A)^\vee \) as in Section 9. But we will not do it here. It is also possible to characterize the relations \( R \in \mathcal{R}_*(A) \) such that \( R^\vee \in \mathcal{R}_*(A) \), as in Section 9 or in terms of a representation \((u_1, u_2)\) of \( R \) (the following result is given without proof):

1. Let \( R \in \mathcal{R}_*^{\infty}(A) \setminus \{0\} \), and let \((u_1, u_2)\) be the representation of \( R \) in \( [0^+, \infty) \). Then \( R^\vee \in \mathcal{R}_*(A) \) if and only if \( u_1 = \infty \) or \( u_2 = 0^+ \).
2. Let \( R \in \mathcal{R}_*(A) \setminus \mathcal{R}_*^{\infty}(A) \), and let \((u_1, u_2)\) be a representation of \( R \). Put \( X = u_1(A) \cap u_2(A) \). Then \( R^\vee \in \mathcal{R}_*(A) \) if and only \( if X \cap [0^+, \infty) = \emptyset \) and \( X \cap \{r^+ : r \in X \cap [0^+, \infty) = \emptyset \}. Moreover, if \( R \in \mathcal{R}_*(A) \) and \((u_1, u_2)\) is a representation in \( \Lambda, then R^\vee \in \mathcal{R}_*(A) \) if and only if \( X \cap [0^+, \infty) = \emptyset \) and \( R^\vee \in \mathcal{R}_*(A) \) if and only if \( X = \emptyset \). *
11. The relation $R_1$ for $R \in \mathcal{R}_\bullet(A)$

For $R \in \mathcal{R}(A)$, we note $R_1$ the binary relation on $A$ defined by (for all $x$, $y \in A$):

$$x R_1 y \Leftrightarrow \Phi_R(x, y) > \Phi_R(y, x).$$

Since for $x$, $y \in A$ and $m$, $n \in \mathbb{N}^*$, we have $\Phi_R(mx, ny) = \frac{m}{n} \Phi_R(x, y)$, $R_1$ is still a positive homothetic order. For $R \in \mathcal{R}(A)$, the indifference relation $S_1 = R_1^\ominus \cap (-R_1)$ associated with $R_1$, is given by (for $x$, $y \in A$)

$$x S_1 y \Leftrightarrow \Phi_R(x, y) = \Phi_R(y, x).$$

In particular, $S_1$ is reflexive. Moreover, we have the

**Proposition 11.1.** Let $R \in \mathcal{R}_\bullet(A)$. Then $R_1$ is a generalized homothetic weak order.

**Proof.** Let $(u_1, u_2)$ be a representation of $R$. We must define a morphism of $\mathbb{N}^\ast$-sets $u : A \rightarrow \mathbb{R}^\geq$ such that for all $x$, $y \in A$, we have $x R_1 y \Leftrightarrow u(x) > u(y)$.

If $R \in\{R_0, R_\infty\}$, then $R_1 = R_\cong$, and the constant functions $u = 0^+$ or $u = \infty$ can be chosen. We can thus suppose that $R \notin\{R_0, R_\infty\}$. By (6.4) and (6.5), we can also suppose that:

- if $R \in \mathcal{R}_\bullet^{0^+, \infty}(A)$, then $(u_1, u_2)$ is the representation of $R$ in $\{0^+, \infty\}$;
- if $R \in \mathcal{R}_\bullet(A) \setminus \mathcal{R}_\bullet^{0^+, \infty}(A)$, then $u_1(A_1^R) \subseteq A$ and $u_2(A) \subseteq A$.

Since $R \notin\{R_0, R_\infty\}$, we have $u_1 \neq 0^+$ and $u_2 \neq \infty$. And we also have:

- if $u_2 = 0^+$, then $R \in \mathcal{R}_\bullet^{0^+, \infty}(A)$ and $u_1(A) = \{0^+, \infty\}$;
- if $u_1 = \infty$, then $R \in \mathcal{R}_\bullet^{0^+, \infty}(A)$ and $u_2(A) = \{0^+, \infty\}$.

If $u_2 = 0^+$, then for $x$, $y \in A$, we have

$$\Phi_R(x, y) = \begin{cases} \infty & \text{if } u_1(x) = \infty \\ 0^+ & \text{if } u_1(x) = 0^+ \end{cases}$$

in particular, $R_1$ is a generalized homothetic weak order represented by the function $u = u_1$. If now $u_1 = \infty$, then for $x$, $y \in A$, we have

$$\Phi_R(x, y) = \begin{cases} \infty & \text{if } u_2(y) = 0^+ \\ 0^+ & \text{if } u_2(y) = \infty \end{cases}$$

in particular, $R_1$ is a generalized homothetic weak order represented by the function $u = u_2^\prime$, defined by $u(x) = u_2(x)^\prime$. We can then suppose that $u_2 \neq 0^+$ and $u_1 \neq \infty$.

By (5), for $(x, y) \in \Phi_R^{-1}(\{0^+, \infty\})$, we have $\Phi_R(x, A_2^R) = \Phi_R(x, y)$ or $\Phi_R(A_1^R, y) = \Phi_R(x, y)$. And for $x$, $y \in A$, by the hypothesis above, we have

$$\Phi_R(x, A_2^R) = \infty \Leftrightarrow u_1(x) = \infty,$$

$$\Phi_R(x, A_2^R) = 0^+ \Leftrightarrow u_1(x) = 0^+,$$

$$\Phi_R(A_1^R, y) = \infty \Leftrightarrow u_2(y) = 0^+,$$

$$\Phi_R(A_1^R, y) = 0^+ \Leftrightarrow u_2(y) = \infty.$$

Recall that $A_{1,2}^R = \{x \in A : u_1(x) \notin \{0^+, \infty\}\}$ and $A_{2,1}^R = \{x \in A : u_2(x) \notin \{0^+, \infty\}\}$. Hence for $x \in A \setminus (A_{1,2}^R \cap A_{2,1}^R) = (A \setminus A_{1,2}^R) \cup (A \setminus A_{2,1}^R)$, there exists $i \in \{1, 2\}$ such that $u_i(x) \in \{0^+, \infty\}$, and we can let

$$u(x) = \begin{cases} \infty & \text{if } u_1(x) = \infty \text{ or } u_2(x) = 0^+ \\ 0^+ & \text{if } u_1(x) = 0^+ \text{ or } u_2(x) = \infty \end{cases}.$$
We must check that for all \( x, y \in A \), we have
\[
\Phi_R(x, y) > \Phi_R(y, x) \iff u(x) > u(y).
\]

Take \( x, y \in A \). If \( x, y \in A_{1,2}^R \cap A_{2,1}^R \), we have
\[
\Phi_R(x, y) > \Phi_R(y, x) \iff \Phi_\triangleright_{\{u_1(x), u_2(y)\}} > \Phi_\triangleright_{\{u_1(y), u_2(x)\}} \\
\iff \gamma^{-1}\cdot x > y \iff u(x) > u(y).
\]

If \( (x, y) \in \Phi_R^{-1}([0^+, \infty)) \) and \( (y, x) \in \Phi_R^{-1}([0^+, \infty)) \), we have
\[
\Phi_R(x, y) > \Phi_R(y, x) \iff \Phi_R(x, A) = \infty \text{ or } \Phi_R(A, y) = \infty \\
\iff u(x) = \infty \text{ or } u(y) = 0^+ \\
\iff u(x) > u(y).
\]

If \( (x, y) \in \Phi_R^{-1}(\emptyset \setminus \{0^+, \infty\}) \) and \( (y, x) \in \Phi_R^{-1}([0^+, \infty)) \), we have
\[
\Phi_R(x, y) > \Phi_R(y, x) \iff \Phi_R(y, A) = 0^+ \text{ or } \Phi_R(A, x) = 0^+ \\
\iff u(y) = 0^+ \text{ or } u(x) = \infty \\
\iff u(x) > u(y).
\]

Finally, if \( (x, y) \in \Phi_R([0^+, \infty)) \), we have
\[
\Phi_R(x, y) > \Phi_R(y, x) \iff \Phi_R(x, y) = \infty \text{ and } \Phi_R(y, x) = 0^+ \\
\iff u(x) = \infty \text{ and } u(y) = 0^+ \\
\iff u(x) > u(y).
\]

This ends the proof of the proposition: \( R_1 \) is a generalized homothetic weak order, represented by \( u \). \( \square \)

To formulate the following results, it is convenient to write:
\[
\begin{align*}
\infty \cdot r & = \infty \quad (r \in \mathbb{R}_{>0}^\geq), \\
0^+ \cdot r & = 0^+ \quad (r \in \mathbb{R}_{>0}^\geq), \\
\infty^{-1} & = 0^+, \\
(0^+)^{-1} & = \infty.
\end{align*}
\]

Beware: we have \( \infty \cdot 0^+ = \infty \text{ and } 0^+ \cdot \infty = 0^+ \).

**Corollary 11.2.** Let \( R \in \mathcal{R}_s(A) \), and let \( (u_1, u_2) \) be a representation of \( R \) such that:
- if \( R = R_0 \), then \( (u_1, u_2) = (0^+, \infty) \);
- if \( R \in \mathcal{R}_s^{0^+, \infty}(A) \), then \( (u_1, u_2) \) is the representation of \( R \) in \( \{0^+, \infty\} \);
- if \( R \in \mathcal{R}_s(A) \setminus \mathcal{R}_s^{0^+, \infty}(A) \), then \( (u_1(A^R_1) \subset A \) and \( u_2(A) \subset A \).

Thus, the function \( u : A \to \mathbb{R}_{>0}^\geq \) defined by
\[
u(x) = \begin{cases} 
\infty & \text{if } u_1(x) = \infty \text{ and } u_2(x) \neq \infty \\
\infty & \text{if } u_1(x) \neq 0^+ \text{ and } u_2(x) = 0^+ \\
0^+ & \text{if } u_1(x) = 0^+ \text{ or } u_2(x) = \infty \\
\gamma^{-1} & \text{if } u_1(x) = \infty \text{ and } u_2(x) \neq \infty \\
\gamma^{-1} & \text{if } u_1(x) \neq 0^+ \text{ and } u_2(x) = 0^+ \\
\gamma^{-1} & \text{if } u_1(x) = 0^+ \text{ or } u_2(x) = \infty \\
r^{1/2} & \text{if } u_2(x) \cdot u_1(x) = r \in \mathbb{R}_{>0}^\geq \\
(r^{1/2})^+ & \text{if } u_2(x) \cdot u_1(x) = r^+ \in \mathbb{R}_{>0}^\geq \setminus A
\end{cases}
\]
is a representation of \( R_1 \). And letting \( \gamma, \gamma^- : A \to A \) be the functions defined by
\[
\gamma(x) = \begin{cases} 
\infty & \text{if } u_1(x) = \infty \text{ and } u_2(x) \neq \infty \\
\infty & \text{if } u_1(x) \neq 0^+ \text{ and } u_2(x) = 0^+ \\
0^+ & \text{if } u_1(x) = 0^+ \text{ or } u_2(x) = \infty \\
[u_2(x)^{-1}u_1(x)]^{1/2} & \text{otherwise}
\end{cases}
\]
and
\[
\gamma^-(x) = \gamma(x)^{-1},
\]
we have \( (\gamma \cdot u, \gamma^- \cdot u) = (u_1, u_2) \).
**Proof.** If \( R = R_{\emptyset} \), then \( u = 0^+ \), \( y = 0^+ \) and \((y \cdot u, y^- \cdot u) = (0^+, \infty) = (u_1, u_2). \) If \( R = R_{\infty} \), then \( u = \infty \), \( y = \infty \) and \((y \cdot u, y^- \cdot u) = (\infty, 0^+) = (u_1, u_2). \) Note that in both cases, \( u \) is a representation of \( R_1 = R_{\emptyset} \). If now \( R \not\in \{R_0, R_{\infty}\} \), then \( u \) is the representation of \( R_1 \) built in the proof of (11.1); and we verify that \((y \cdot u, y^- \cdot u) = (u_1, u_2). \) \( \square \)

Let \( A/\mathbb{N}^* \) be the quotient-set of \( A \) by the equivalence relation \( \sim_{\mathbb{N}^*} \) on \( A \) defined by \( x \sim_{\mathbb{N}^*} y \Leftrightarrow \) there exist \( m, n \in \mathbb{N}^* \) such that \( mx = ny. \)

**Corollary 11.3.** Let \( R \in \mathcal{R}_+(A). \) There exist a morphism of \( \mathbb{N}^* \)-sets \( u : A \rightarrow \mathbb{R}^+_{\geq 0} \) and a map \( \gamma : A/\mathbb{N}^* \rightarrow A \) such that (for all \( x, y \in A \))

i. \( u(A^+_1) \subseteq A, \)

ii. \( \gamma^{-1}(\infty) = u^{-1}(\infty), \)

iii. \( \gamma^{-1}(0^+) = u^{-1}(0^+), \)

iv. \( x \cdot x y \Leftrightarrow \gamma(x) \cdot u(x) > \gamma(y)^{-1} \cdot u(y). \)

Moreover, up to multiplication by an element of \( \mathbb{R}_{>0}, \) the pair \( (u, \gamma) \) is unique: if \( (u', \gamma') \) is another pair of maps like above and satisfying the conditions (i), (ii), (iii), (iv), then there exists a \( \lambda \in \mathbb{R}_{>0} \) such that \( (u', \gamma') = (\lambda \cdot u, \gamma). \)

**Proof.** The existence of the pair \( (u, \gamma) \) results from the **Corollary 11.2**; note that by construction, \( u : A \rightarrow \mathbb{R}^+_{\geq 0} \) is a morphism of \( \mathbb{N}^* \)-sets, and \( \gamma : A \rightarrow A \) factors through \( A/\mathbb{N}^*. \) The uniqueness of the pair \( (u, \gamma) \) is a consequence of the uniqueness property in **Lemmas 6.4 and 6.5**. \( \square \)

**Corollary 11.4.** Let \( R \in \mathcal{R}_+(A) \), and let \( u : A \rightarrow \mathbb{R}^+_{\geq 0} \) and \( \gamma : A/\mathbb{N}^* \rightarrow A \) be a morphism of \( \mathbb{N}^* \)-sets and a map, satisfying the conditions (i), (ii), (iii), (iv) of (11.3). Then \( u \) represents \( R_1. \)

**Remark 11.5.** For \( R \in \mathcal{R}(A) \setminus \mathcal{R}_+(A), \) the relation \( R_1 \) is not always a generalized homothetic weak order. We can for instance verify that the relation \( R \) of the **Remark 9.5** satisfies \( R_1 = R. \) \( \star \)

**Remark 11.6.** For \( R \in \mathcal{R}(A) \) and \( n \in \mathbb{N}^*, \) we define by induction an order \( R_{n+1} \in \mathcal{R}(A): \) we put \( R_{n+1} = (R_n)_1. \) For all \( R \in \mathcal{R}(A) \) and all \( n \in \mathbb{N}^*, \) one can verify that \( R_n = R_1. \) \( \star \)

### 12. Comments

The **Corollary 11.3** is a generalization of [13]. Indeed, in [13] we have obtained the same result but only for homothetic interval orders on \( A. \) Representing a relation \( R \in \mathcal{R}_+(A) \) by a pair \( (u, \gamma) \) as in (11.3) rather than by a pair \( (u_1, u_2) \) like in (6.4) and (6.5), has the advantage of showing the underlying generalized homothetic weak order \( R_1 \) (represented by \( u \)). We can then “see” \( R \) as a deformation of \( R_1, \) the deformation being represented by the twisting factor \( \gamma : A \rightarrow A. \) This naturally leads to group in a single family the relations \( R \in \mathcal{R}_+(A) \) having the same underlying generalized homothetic weak order \( R_1. \)

The introduction of the set \( \mathbb{R}^+_{\geq 0} \) is not merely an ad hoc construction to treat the abandon of the super-Archimedean property. Recall that \( \mathbb{R}^+_{\geq 0} \) is the set of intervals of \( \mathbb{R}_{>0} \) of the form \([r, +\infty[ \) and \( ]r, \infty[ \), to which the empty interval is added. The name itself of “interval order” naturally leads to the following question: why limiting oneself to relations that can be represented by closed intervals, and not consider the relations that can be by intervals which are closed or open. The set \( \mathbb{R}^+_{\geq 0} \) is a response to this question. Another response is given by the following variant of **Lemma 6.5**:

**Lemma 12.1.** Let \( R \in \mathcal{R}_+(A) \setminus \mathcal{R}_+^{0, \infty}(A). \) There exist two morphisms of \( \mathbb{N}^* \)-sets \( v_1, v_2 : A \rightarrow A \) such that for all \( x, y \in A, \) we have

\[ x R y \Leftrightarrow \begin{cases} v_1(x) > v_2(y), & \text{if } (x, y) \in A^{R} \\ v_1(x) \geq v_2(y), & \text{if } (x, y) \in B^{R}. \end{cases} \]

Moreover, up to multiplication by an element of \( \mathbb{R}_{>0}, \) the pair \( (v_1, v_2) \) is unique.

**Proof.** By (6.5), there exists a representation \((u_1, u_2)\) of \( R \) such that \( u_1(A^+_1) \subseteq A \) and \( u_2(A) \subseteq A. \) Consider the projection \( \mathbb{R}^+_{>0} \rightarrow A \rightarrow \tilde{r} \) defined in the proof of (9.3). And for any function \( u : A \rightarrow \mathbb{R}^+_{>0}, \) note \( \tilde{u} : A \rightarrow A \) the function defined by \( \tilde{u}(x) = u(x). \) Then, the pair \( (v_1, v_2) = (\tilde{u}_1, u_2) \) satisfies the conditions of the lemma. And the uniqueness property of \((v_1, v_2)\) results from the uniqueness property of \((u_1, u_2). \) \( \square \)

In our opinion, the answer (6.5) is preferable to the answer (12.1). Indeed, in (12.1), we must first choose whether a triplet \((x, R, y)\) is or is not super-Archimedean before being able to decide whether \( x R y \) or \( x \not R y \) with the pair of functions \((v_1, v_2). \) On the other hand, in (6.5), the fact that a triplet \((x, R, y)\) is or is not super-Archimedean is deduced a posteriori from the representation \((u_1, u_2); \) i.e. the pair of values \((u_1(x), u_2(y)) \in \mathbb{R}^+_{>0} \times \mathbb{R}^+_{>0} \) allows not only deciding if \( x R y \) or \( x \not R y, \) but also deciding whether \((x, R, y)\) is super-Archimedean or not.

The study of positive homothetic orders on \( A \) which are not generalized homothetic bidders will be the focus of a further work.
References